Data-Driven Minimum-Energy Controls for Linear Systems

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***Abstract*—In this letter, we study the problem of computing minimum-energy controls for linear systems from experimental data. The design of open-loop minimum-energy control inputs to steer a linear system between two different states in finite time is a classic problem in control theory, whose solution can be computed in closed form using the system matrices and its controllability Gramian. Yet, the computation of these inputs is known to be ill-conditioned, especially when the system is large, the control horizon long, and the system model uncertain. Due to these limitations, open-loop minimum-energy controls and the associated state trajectories have remained primarily of theoretical value. Surprisingly, in this letter, we show that open-loop minimum-energy controls can be learned exactly from experimental data, with a finite number of control experiments over the same time horizon, without knowledge or estimation of the system model, and with an algorithm that is significantly more reliable than the direct model-based computation. These findings promote a new philosophy of controlling large, uncertain, linear systems where data is abundantly available.**

***Index Terms*—Linear systems, optimal control, statistical learning, identification for control, control of networks.**

I. Introduction

CONSIDER the discrete-time linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where, respectively, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ denote the system and input matrices, and $x : \mathbb{N} \to \mathbb{R}^n$ and $u : \mathbb{N} \to \mathbb{R}^m$ describe the state and input of the system. For a control horizon $T \in \mathbb{N}$ and a desired state x_f, the minimum-energy control problem asks for the input sequence $u(0), \dots, u(T-1)$ with minimum energy that steers the state from x_0 to x_f in T steps, and it can be formulated as

$$\begin{aligned} \min_u \ & \sum_{t=0}^{T-1} \|u(t)\|_2^2, \\ \text{s.t.} \ & x(t+1) = Ax(t) + Bu(t), \\ & x(0) = x_0, \ x(T) = x_\mathrm{f}. \end{aligned} \tag{2}$$

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As a classic result [1], the minimization problem (2) is feasible if and only if $(x_\mathrm{f} - A^T x_0) \in \mathrm{Im}(W_T)$, where

$$W_T = \sum_{t=0}^{T-1} A^t B B^\mathsf{T} (A^\mathsf{T})^t \tag{3}$$

is the T-steps controllability Gramian and $\mathrm{Im}(W_T)$ denotes the image of the matrix W_T. Further, the solution to (2) is

$$u^*(t) = B^\mathsf{T} (A^\mathsf{T})^{T-t-1} W_T^\dagger (x_\mathrm{f} - A^T x_0), \tag{4}$$

where $W_T^\dagger$ is the Moore–Penrose pseudoinverse of W_T [2].

The controllability Gramian (3) and the minimum-energy control input (4) identify fundamental control limitations for the system (1), and have been extensively used to solve design [3], sensor and actuator placement [4], and control problems [5] for systems and networks. However, besides their theoretical value, the optimal control input (4) is rarely used in practice or even computed numerically because (i) it relies on the perfect knowledge of the system dynamics, (ii) its performance is not robust to model uncertainties, and (iii) the controllability Gramian is typically ill-conditioned, especially when the system is large [5], [6]. This implies that the control sequence (4) is numerically difficult to compute, and that its implementation leads to errors [7]. To the best of our knowledge, efficient and numerically reliable methods to compute minimum-energy control inputs are still lacking.

Paper Contributions: This letter features two main contributions. First, we show that minimum-energy control inputs for linear systems can be computed from data obtained from control experiments with non-minimum-energy inputs, and without knowledge or estimation of the system matrices. Thus, optimal inputs can be learned from *non-optimal* ones, and we provide three different expressions for doing so. Surprisingly, we also establish that a *finite* number of non-optimal control experiments is always sufficient to compute minimum-energy control inputs towards any reachable state. Second, we show that the data-driven computation of minimum-energy inputs is numerically as reliable as the computation of the inputs based on the exact knowledge of the system matrices, and substantially more reliable than using the closed-form expression based on the Gramian. Further, as minor contributions, we (i) derive bounds on the number of required control experiments as a function of the dimension of the system, number of control inputs, and length of the control horizon, (ii) discuss the effect of noisy data on the data-driven expressions, and (iii) extend our data-driven framework to the case of output measurements.

Our results suggest the tantalizing hypothesis that several optimal control problems can be solved efficiently and reliably using a combination of data-driven algorithms and system properties (in our setup, linearity of the dynamics), even when the system model is uncertain or unknown.

Related Work: Several works investigate the problem of estimating optimal controls for linear systems from input-output data. The classic model-based approach [8] consists of (i) identifying a model of the system from the available data, and (ii) using the estimated model to design the optimal control inputs. Data-driven algorithms have been proposed in [9]–[12] for the LQR/LQG problem. In particular, the approach pursued in these papers relies on the estimation of the Markov parameters of the system, thereby bypassing the identification step of the model-based approach. Differently from the above approaches, in this letter we focus on computing open-loop minimum-energy inputs from experimental data, without reconstructing the system matrices and where the experiments use arbitrary control inputs. To the best of our knowledge, this letter addresses a novel problem and provides new and numerically more reliable expressions for the computation of minimum-energy control inputs.

II. Learning Minimum-Energy Control Inputs

In vector form, the minimum-energy control problem asks to find the minimum-norm solution to the following equation:

$$x_{\mathrm{f}} = A^T x_0 + \underbrace{\begin{bmatrix} B & AB & \cdots & A^{T-1}B \end{bmatrix}}_{C_T} u,$$

where the vector $u \in \mathbb{R}^{mT}$ contains the control inputs over the control horizon $[0, T-1]$, namely $u = [u(T-1)^{\mathsf{T}} \ \ldots \ u(0)^{\mathsf{T}}]^{\mathsf{T}}$, and C_T denotes the T-steps controllability matrix.[1] Then, if the controllability matrix C_T is known, the minimum-energy control input to reach x_{f} is

$$u^* = C_T^{\dagger}(x_{\mathrm{f}} - A^T x_0). \tag{5}$$

Instead of using (5), in this letter we aim to compute minimum-energy control inputs leveraging a set of N control experiments and assuming that the system matrices, and thus the controllability matrix, are not available. The i-th control experiment consists of applying the input sequence u_i to (1), and measuring the system state at time T, namely x_i, where

$$x_i = A^T x_0 + C_T u_i. \tag{6}$$

We remark that the inputs u_i are arbitrary and not necessarily of minimum-norm. In vector form, the available data is

$$X = \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix}, \ \text{and} \ \ U = \begin{bmatrix} u_1 & \cdots & u_N \end{bmatrix}, \tag{7}$$

where x_i is the state at time T with input u_i as in (6).[2]

A. Data-Driven Minimum-Energy Controls

Because we only rely on the experimental data (X, U) to learn the minimum-energy control input to reach a desired state, we postulate that such input can be computed as a linear combination of the inputs U. Thus, we formulate and study the following constrained minimization problem:

$$\begin{aligned} \alpha^* = \arg\min_{\alpha} \quad & \|U\alpha\|_2^2, \\ \text{s.t.} \quad & x_{\mathrm{f}} = X\alpha, \end{aligned} \tag{8}$$

where $\alpha \in \mathbb{R}^N$ is the optimization variable. As we show in Theorem 1, a first data-driven expression for the minimum-energy control input derives from a solution to (8). We start with the expression of the minimum-energy control input for the case $x_0 = 0$, and we postpone the general case $x_0 \neq 0$ to Remark 2. Let $\mathrm{Im}(M)$ and $\mathrm{Ker}(M)$ denote the range-space and the null-space of the matrix M, respectively. With a slight abuse of notation, we write $K = \mathrm{Im}(A)$ (resp. $K = \mathrm{Ker}(A)$) to say that K is a basis of $\mathrm{Im}(A)$ (resp. $\mathrm{Ker}(A)$). A matrix is full row rank if the dimension of its range-space equals the number of its rows.

Theorem 1 (Data-Driven Minimum-Energy Control Inputs When $x_0 = 0$): If the matrix U in (7) is full row rank, then, for any final state x_{f}, the minimum-energy input equals

$$u^* = (I - UK(UK)^{\dagger})UX^{\dagger}x_{\mathrm{f}}, \tag{9}$$

where $K = \mathrm{Ker}(X)$ and X is as in (7).

Proof: We first show that (8) is feasible, and that $u^* = U\alpha^*$. Notice that, because U is full row rank, there exists α^* such that $u^* = U\alpha^*$, where u^* is the minimum-energy control input to reach x_{f}. Additionally, α^* satisfies the constraint in (8) because $X\alpha^* = C_T U\alpha^* = C_T u^* = x_{\mathrm{f}}$. Finally, because u^* is unique [1], α^* is also a solution to (8), and its computation is equivalent to computing the input u^*.

To compute α^* we solve the constraint $x_{\mathrm{f}} = X\alpha$ and substitute it in the cost function. Namely, $\alpha = X^{\dagger}x_{\mathrm{f}} - Kw$, where $K = \mathrm{Ker}(X)$ and w is an arbitrary vector. Equating to zero the derivative of the cost function with respect to w, we obtain $w^* = (UK)^{\dagger}UX^{\dagger}x_{\mathrm{f}}$. This implies that $\alpha^* = X^{\dagger}x_{\mathrm{f}} - Kw^*$, from which (9) follows by letting $u^* = U\alpha^*$. ■

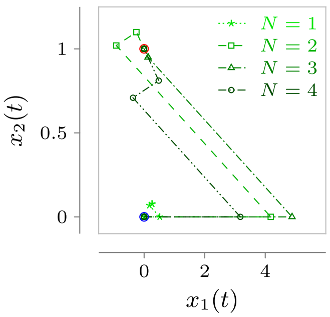
Theorem 1 provides an expression of the minimum-energy control input, which only uses data originated from a set of control experiments, and does not require the knowledge of the system matrices. Importantly, Theorem 1 shows that minimum-energy control inputs can be directly computed based on a number of control experiments with arbitrary, thus not minimum-energy, inputs. Further, Theorem 1 assumes that U is full row rank, which guarantees the computation of the minimum-energy input for any final state x_{f}. When U is not full row rank but $u^* \in \mathrm{Im}(U)$, the minimum-energy control input can still be computed as in Theorem 1. Instead, when $u^* \notin \mathrm{Im}(U)$, the minimum-energy input cannot be computed as a (linear) combination of the experimental data (7). In this case, the data-driven input (9) reaches the desired final state x_{f}, if $x_{\mathrm{f}} \in \mathrm{Im}(X)$, or the final state $\tilde{x}_{\mathrm{f}} \in \mathrm{Im}(X)$ that is closest to x_{f}, if $x_{\mathrm{f}} \notin \mathrm{Im}(X)$. To see this, let u^* be as in (9) and note that

$$\begin{aligned} \tilde{x}_{\mathrm{f}} &= C_T u^* = C_T(I - UK(UK)^{\dagger})UX^{\dagger}x_{\mathrm{f}} \\ &= C_T UX^{\dagger}x_{\mathrm{f}} - \underbrace{C_T UK(UK)^{\dagger}UX^{\dagger}x_{\mathrm{f}}}_{=0 \text{ because } C_T UK = XK = 0} = XX^{\dagger}x_{\mathrm{f}}, \end{aligned}$$

which shows that $\tilde{x}_{\mathrm{f}}$ is the orthogonal projection of x_{f} onto $\mathrm{Im}(X)$. This in particular implies that the error $\|x_{\mathrm{f}} - \tilde{x}_{\mathrm{f}}\|_2$ is non-increasing in the number of experiments N, and it vanishes when the experimental data satisfies $x_{\mathrm{f}} \in \mathrm{Im}(X)$. Finally,

[1] To simplify the technical treatment and without compromising generality, we assume that x_{f} is reachable in T-steps, i.e., $(x_{\mathrm{f}} - A^T x_0) \in \mathrm{Im}(C_T)$.

[2] While the full state trajectory could be measured [13], here we show that measuring the final state is sufficient to compute minimum-energy inputs.



(a)

$$A = \begin{bmatrix} -0.5 & 0 \\ 0.1 & 0.8 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b)

$$u_1 = \begin{bmatrix} -1.138 \\ -1.507 \\ -0.749 \\ 0.101 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1.625 \\ -1.916 \\ 1.154 \\ 1.307 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -1.096 \\ -0.050 \\ -0.147 \\ -0.886 \end{bmatrix} \quad u_4 = \begin{bmatrix} 0.906 \\ 0.620 \\ 0.089 \\ -2.328 \end{bmatrix}$$

(c)

$$\|u^*\|_2 = 9.317$$

$$\|\hat{u}\|_2 = 0.924 \quad \|\hat{u}\|_2 = 11.52$$

$$\|\hat{u}\|_2 = 10.73 \quad \|\hat{u}\|_2 = 9.317$$

(d)

Fig. 1. Panel (a) illustrates the state trajectory $x(t) = [x_1(t)\ x_2(t)]^\mathsf{T}$, $t \in [0, T]$, of the system described by matrices A, B as in (b) and driven by the data-driven input $\hat{u}$ (9), for four values of the number of experiments N. We choose $T = 4$ and denote the initial state $x_0 = [0\ 0]^\mathsf{T}$ and final state $x_\mathrm{f} = [0\ 1]^\mathsf{T}$ with a blue and red circle, respectively. The other markers correspond to the values of the trajectories in the interval $[0, T]$. The data-driven input $\hat{u}$ has been computed using input data u_i as in (c), and $x_i = C_T u_i$. Panel (d) shows the norm of the minimum-energy input and the norm of the data-driven input $\hat{u}$ (9) as N varies (color-coded).

Theorem 1 can be used to quantify the number of experiments needed to compute minimum-energy inputs.

Corollary 1 (Required Number of Control Experiments to Compute Minimum-Energy Inputs): Let n be the dimension of the system, m the number of inputs, T the control horizon, and N the number of control experiments. Then,

(i) $N \geq n$ is necessary to compute minimum-energy control inputs towards any arbitrary final state x_f;

(ii) $N = mT$ is sufficient to compute minimum-energy control inputs towards any arbitrary final state x_f, provided that the inputs u_i are linearly independent.

Proof: (Necessity) Assume by contradiction that the number of experiments is strictly less than n. Then, $\mathrm{Rank}(X) < n$, and there exists $x_\mathrm{f} \notin \mathrm{Im}(X)$. Then, the minimization problem (8) is infeasible, and the minimum-energy control input cannot be computed from the inputs U.

(Sufficiency) Let the experimental inputs be linearly independent. Then, U is invertible and, for any x_f, there exists a solution α^* such that $u^* = U\alpha^*$. This shows that the minimum-energy input can be computed from the data. ■

Corollary 1 characterizes the number of control experiments that are required to compute minimum-energy control inputs from experimental data. In particular, as few as n experiments are needed, in which case the experiments must contain n linearly independent minimum-energy control inputs, and as many as mT experiments are sufficient, in which case the control inputs can be selected arbitrarily provided that they form a linearly independent set of vectors. This also shows that optimal control inputs can be learned from a *finite* number of non-optimal control inputs.

Example 1 (Data-Driven Control Inputs When $N \leq mT$): We consider a two-dimensional system with matrices A and B as in Fig. 1(b), control horizon $T = 4$, initial state $x_0 = [0\ 0]^\mathsf{T}$, and final state $x_\mathrm{f} = [0\ 1]^\mathsf{T}$. We vary the number of control experiments N from 1 to 4, where the inputs are as in Fig. 1(c). For each number of control experiments, we compute the data-driven input (9), and report the corresponding state trajectory and norm in Fig. 1(a) and Fig. 1(d), respectively. Notice that, when $N = 1$, the data-driven input does not steer the system state to x_f. Instead, for $N = 2, 3, 4$ the state trajectory reaches x_f. Finally, the data-driven input has minimum norm only when $N = 4$.

Remark 1 (Geometric Properties of (9)): Several geometric properties of (9) can be highlighted. First, $UK = \mathrm{Ker}(C_T)$ when U is full row rank. In fact, $C_T UK = XK = 0$, showing that $\mathrm{Im}(UK) \subseteq \mathrm{Ker}(C_T)$. Further, if $C_T u = 0$ and $u = U\alpha$, then, $X\alpha = C_T U\alpha = C_T u = 0$, showing that $\alpha \in \mathrm{Im}(K)$ and $\mathrm{Ker}(C_T) \subseteq \mathrm{Im}(UK)$. Thus, $\mathrm{Im}(UK) = \mathrm{Ker}(C_T)$ when U is full row rank. Second, $I - UK(UK)^\dagger$ is the orthogonal projection onto the kernel of $(UK)^\mathsf{T}$ and, consequently, $u^* = (I - UK(UK)^\dagger)UX^\dagger x_\mathrm{f}$ is orthogonal to $\mathrm{Ker}(C_T)$. This is expected, because u^* is the minimum-energy input to reach the state x_f.

B. Alternative Expression of Minimum-Energy Controls

In this subsection, we present a different optimization problem that can be used to derive an equivalent expression of the data-driven minimum-energy control input (9). Specifically, we consider the following problem, which encodes the problem of estimating the controllability matrix from data:

$$C_T^* = \arg\min_C \|X - CU\|_F^2, \tag{10}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. The above problem has a unique solution, which equals $C_T^* = XU^\dagger$. Notice that the minimization problem (10) returns an estimate of the controllability matrix, which can be used to compute the input as $\hat{u} = (C_T^*)^\dagger x_\mathrm{f} = (XU^\dagger)^\dagger x_\mathrm{f}$. We next show that $\hat{u}$ coincides with the control input (9).

Theorem 2 (Equivalent Expressions of Data-Driven Minimum-Energy Inputs): Let X and U be as in (7). Then,

$$(I - UK(UK)^\dagger)UX^\dagger x_\mathrm{f} = (XU^\dagger)^\dagger x_\mathrm{f}. \tag{11}$$

Proof: We show that $(XU^\dagger)^\dagger = (I - UK(UK)^\dagger)UX^\dagger$. That is, we show that $(I - UK(UK)^\dagger)UX^\dagger$ satisfies the four conditions [2] defining the Moore–Penrose pseudoinverse of $XU^\dagger$. To this aim, let $K = I - X^\dagger X$. Since $P = I - UK(UK)^\dagger$ is the orthogonal projection onto $\mathrm{Ker}((UK)^\mathsf{T})$,

$$(UK)^\mathsf{T} P = 0 \overset{P = P^\mathsf{T}}{\Longrightarrow} PUK = 0 \Rightarrow PUX^\dagger X = PU. \tag{12}$$

Because $X = C_T U$, we have $\mathrm{Ker}(U) \subseteq \mathrm{Ker}(X)$. Since $I - U^\dagger U$ is the orthogonal projection onto $\mathrm{Ker}(U)$, we have

$$X(I - U^\dagger U) = 0 \Rightarrow XU^\dagger U = X. \tag{13}$$

Further, using $XK = 0$, we obtain

$$XU^\dagger(I - P) = XU^\dagger UK(UK)^\dagger \overset{(13)}{=} XK(UK)^\dagger = 0. \tag{14}$$

Finally, since $I - UU^\dagger$ denotes the orthogonal projection onto $\mathrm{Ker}(U^\mathsf{T})$, and $UK(UK)^\dagger$ the orthogonal projection onto $\mathrm{Im}(UK) \subseteq \mathrm{Im}(U) \perp \mathrm{Ker}(U^\mathsf{T})$, we have

$$\begin{aligned} & UK(UK)^\dagger = I - P = 0 \\ & \Rightarrow\ (I - P)(I - UU^\dagger) = [(I - P)(I - UU^\dagger)]^\mathsf{T} \\ & \Rightarrow\ UU^\dagger P = PUU^\dagger, \end{aligned} \tag{15}$$

where the last implication follows because $I - P$ and $I - UU^\dagger$ are symmetric. To conclude, we show that $PUX^\dagger = (XU^\dagger)^\dagger$ by proving the four Moore–Penrose conditions [2]:

(i) $PUX^\dagger XU^\dagger PUX^\dagger \overset{(12)}{=} PUU^\dagger PUX^\dagger \overset{(15)}{=} P^2 UU^\dagger \cdot UX^\dagger = PUX^\dagger$;

(ii) $XU^\dagger PUX^\dagger XU^\dagger \overset{(12)}{=} XU^\dagger PUU^\dagger = XU^\dagger UU^\dagger - XU^\dagger (I - P)UU^\dagger \overset{(14)}{=} XU^\dagger$;

(iii) $XU^\dagger PUX^\dagger = XU^\dagger UX^\dagger - XU^\dagger (I - P)UX^\dagger \overset{(13),(14)}{=} XX^\dagger = (XX^\dagger)^\mathsf{T}$;

(iv) $PUX^\dagger XU^\dagger \overset{(12)}{=} PUU^\dagger \overset{(15)}{=} UU^\dagger P = (PUU^\dagger)^\mathsf{T}$. ■

C. Asymptotic Expression of Minimum-Energy Controls

The minimization problem (10) reconstructs the forward controllability matrix C_T, from which minimum-energy control inputs can be derived by subsequently computing $C_T^\dagger$. To avoid the computation of $C_T^\dagger$ and obtain a potentially simpler expression, we next consider the problem of directly estimating $C_T^\dagger$ from the experimental data:

$$M^* = \arg\min_M \|MX - U\|_F^2. \tag{16}$$

Notice that the latter problem is equivalent to estimating the inverse map from X to U, and it is typically more difficult than the problem of estimating the map from U to X. In fact, while the forward map is unique, the inverse map is typically not.[3] Further, the control input $M^* x_\mathrm{f}$ obtained by solving the minimization problem (16) is not guaranteed to be of minimum norm and to steer the system to x_f, as these constraints do not appear in the minimization problem. In what follows, we say that a sequence of random matrices $\{X_n\}_{n\in\mathbb{N}}$ converges almost surely (a.s.) to a matrix X, and denote it with $X_n \xrightarrow{\text{a.s.}} X$, if $\Pr(\lim_{n\to\infty} X_n = X) = 1$.

Theorem 3 (Asymptotically Equivalent Expression to (9)): Let X and U be as in (7). The unique solution to the minimization problem (16) is

$$M^* = UX^\dagger, \tag{17}$$

and the corresponding control input can be written as

$$\hat{u} = M^* x_\mathrm{f} = UX^\dagger x_\mathrm{f}. \tag{18}$$

Further, if X is full row rank, then $C_T M^* x_\mathrm{f} = x_\mathrm{f}$. That is, the control $\hat{u}$ steers the system from $x_0 = 0$ to $x(T) = x_\mathrm{f}$. Finally, if the entries of U are i.i.d. random variables with zero mean and nonzero finite variance, then $UX^\dagger \xrightarrow{\text{a.s.}} C_T^\dagger$ as $N \to \infty$. That is, as the number of control experiments increases, the input $\hat{u}$ converges a.s. to the optimal input u^*.

Proof: The expression (17) follows from the properties of the Moore–Penrose pseudoinverse. For the second claim, we note that $C_T \hat{u} = C_T UX^\dagger x_\mathrm{f} = XX^\dagger x_\mathrm{f} = x_\mathrm{f}$, where we have used that X is full row rank and $X = C_T U$. To prove the third statement, let $N \to \infty$, and let the control experiments be chosen so that the entries of U are i.i.d. random variables with zero mean and finite variance σ^2. Let U_{ij} denote the (i,j)-th entry of U, and observe that the (i,j)-th entry of $\frac{1}{N}UU^\mathsf{T}$ equals $\frac{1}{N}\sum_{k=1}^N U_{ik}U_{jk}$. Because $\{U_{ik}U_{jk}\}_{k\in\mathbb{N}}$ is an i.i.d. sequence of random variables, for all $i, j \in \{1, \dots, N\}$ and, due to the Strong Law of Large Numbers [14, p. 6], when $N \to \infty$ we have

$$\frac{1}{N}\sum_{k=1}^N U_{ik}U_{jk} \xrightarrow{\text{a.s.}} \mathbb{E}[U_{i1}U_{j1}] = \begin{cases} \sigma^2, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\mathbb{E}[\,\cdot\,]$ denotes the expected value operator. Then,

$$\frac{1}{N}UU^\mathsf{T} \xrightarrow{\text{a.s.}} \sigma^2 I \quad \text{as } N \to \infty. \tag{19}$$

Next, consider the function $f : \mathbb{R}^{mT\times mT} \to \mathbb{R}^{mT\times n}$, $Y \mapsto YC_T^\mathsf{T}(C_T Y C_T^\mathsf{T})^\dagger$. Note that $f(Y)$ is continuous at $Y = \alpha I$, $\alpha > 0$,[4] and $f(\alpha I) = C_T^\mathsf{T}(C_T C_T^\mathsf{T})^\dagger = C_T^\dagger$ [2, p. 49]. To conclude, we employ the Continuous Mapping Theorem [14, Th. 2.3(iii)] and (19) to obtain, as $N \to \infty$,

$$\begin{aligned} UX^\dagger &= U(C_T U)^\dagger = \frac{1}{N}UU^\mathsf{T}C_T^\mathsf{T}\left(C_T \frac{1}{N}UU^\mathsf{T}C_T^\mathsf{T}\right)^\dagger \\ &= f\left(\frac{1}{N}UU^\mathsf{T}\right) \xrightarrow{\text{a.s.}} f\left(\sigma^2 I\right) = C_T^\dagger. \end{aligned}$$

■

Theorem 3 contains a data-driven expression of the minimum-energy control input for a linear system, which does not rely on the estimation of the system matrices or the controllability matrix. As we show in the next section, the expression (18) is not only conceptually simpler than the classic Gramian-based expression of the minimum-energy control input and our other data-driven expressions (9) and (11), but it is also numerically more reliable as it requires a smaller number of operations. Yet, differently from (9) and (11), the expression (18) coincides with the minimum-energy control only asymptotically in the number of experiments, and assuming that the entries of the input matrix U are zero-mean i.i.d. random variables with nonzero finite variance.

Remark 2 (Data-Driven Minimum-Energy Control Inputs When $x_0 \neq 0$): When $x_0 \neq 0$, the computation of the minimum-energy control input to reach x_f is more involved, as the unknown matrix A and vector x_0 enter the relation (6).[5] Yet, under a mild assumption on the experimental inputs U, minimum-energy inputs can still be computed with a finite number of experiments. To see this, consider the problem

$$\begin{aligned} \min_\alpha \quad & \|U\alpha\|_2^2, \\ \text{s.t.} \quad & x_\mathrm{f} = X\alpha \quad \text{and} \quad 1 = \mathbb{1}^\mathsf{T}\alpha, \end{aligned} \tag{20}$$

Assume that the matrix U is full row rank, and that there exists a vector w such that $Uw = 0$ and $\mathbb{1}^\mathsf{T}w \neq 0$. The first assumption guarantees that there exists α^* such that $u^* = U\alpha^*$, and thus the computation of the minimum-energy control for any final state x_f (see Theorem 2.1). The second assumption ensures that there exists α^* satisfying $1 = \mathbb{1}^\mathsf{T}\alpha^*$, which allows us to correctly reconstruct the term $A^T x_0$ from X.[6] In fact, let $\alpha^* = U^\dagger u^* + w(1 - \mathbb{1}^\mathsf{T}U^\dagger u^*)/(\mathbb{1}^\mathsf{T}w)$, and notice that $u^* = U\alpha^*$, where u^* is the minimum-energy control

[3] In particular, the inverse map is not unique whenever $mT > n$.

[4] In fact, since $\mathrm{Rank}(C_T Y C_T^\mathsf{T}) = \mathrm{Rank}(C_T C_T^\mathsf{T})$ for any positive definite Y, it holds $\lim_{k\to\infty}(C_T Y_k C_T^\mathsf{T})^\dagger = (\alpha\, C_T C_T^\mathsf{T})^\dagger$ for any sequence of positive definite matrices $\{Y_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} Y_k = \alpha I$ [2, p. 238].

[5] Notice that the term $A^T x_0$ remains unknown even if the exact value of $x_0 \neq 0$ is known. Thus knowledge of x_0 does not modify the expressions we obtain when $x_0 \neq 0$ is treated as an unknown variable.

[6] These assumptions can always be satisfied by properly designing the experimental inputs, or by running sufficiently many random experiments.

input to reach x_{f}. Further, using (6) and $1 = \mathbb{1}^{\mathsf{T}}\alpha^*$, we have $X\alpha^* = \sum_{i=1}^{N} X_i\alpha_i^* = A^T x_0 \sum_{i=1}^{N} \alpha_i^* + C_T \sum_{i=1}^{N} \alpha_i^* U_i = A^T x_0 + C_T u^* = x_{\mathrm{f}}$. Then, similarly to the proof of Theorem 1, a solution to (20) determines the minimum-energy input.

To solve the minimization problem (20), let $\bar{X} = [X^{\mathsf{T}}\ \mathbb{1}]^{\mathsf{T}}$ and $\bar{x}_{\mathrm{f}} = [x_{\mathrm{f}}^{\mathsf{T}}\ 1]^{\mathsf{T}}$. Then, similarly to Theorem 1, we obtain $\alpha^* = \bar{X}^\dagger x_{\mathrm{f}} - K(UK)^\dagger U\bar{X}^\dagger \bar{x}_{\mathrm{f}}$, where $K = \mathrm{Ker}(\bar{X})$, and

$$u^* = (I - UK(UK)^\dagger)U\bar{X}^\dagger \bar{x}_{\mathrm{f}}. \tag{21}$$

Because the matrix U is required to have a nontrivial null-space, a sufficient number of linearly-independent non-optimal experiments for the computation of the minimum-energy control input to any arbitrary final state is $mT + 1$.

Finally, from the above reasoning and the proof of Theorem 2 and Theorem 3, the minimum-energy input (21) can be written equivalently as $u^* = (\bar{X}U^\dagger)^\dagger \bar{x}_{\mathrm{f}} = U\bar{X}^\dagger \bar{x}_{\mathrm{f}}$, where the last equality holds asymptotically for any choice of inputs satisfying the assumptions in Theorem 3.

Remark 3 (Data-Driven Expressions With Noisy Data): Let the measurements of the input u_i and the final state x_i be corrupted by noise. Let $\tilde{U} = [u_1 + w_1\ \cdots\ u_N + w_N]$ and $\tilde{X} = [x_1 + v_1\ \cdots\ x_N + v_N]$ be the matrices obtained by concatenating all noisy measurements. The data-driven estimates (9), (11), and (18) computed from the noisy data $(\tilde{U}, \tilde{X})$ are typically biased. To see this, consider the system $x(t+1) = ax(t) + u(t)$, $a \in \mathbb{R}$, $x_0 = 0$, and $T = N = 1$. In this simple case, expressions (9), (11), and (18) are equivalent and, assuming that $x_1 + v_1 \neq 0$, read as $\hat{u} = \frac{u_1 + w_1}{x_1 + v_1} x_{\mathrm{f}}$. If w_1 and v_1 are independent random variables uniformly distributed in $[-\varepsilon, \varepsilon]$, with $0 < \varepsilon < |u_1|$, it holds

$$\begin{aligned}\mathrm{Bias}[\hat{u}] &= \mathbb{E}_{w_1,v_1}[\hat{u}] - u^* = \mathbb{E}_{v_1}\left[\frac{u_1}{u_1 + v_1}\right] x_{\mathrm{f}} - x_{\mathrm{f}} \\ &= \left[\frac{1}{2\varepsilon} u_1 \ln\left(\frac{u_1 + \varepsilon}{u_1 - \varepsilon}\right) - 1\right] x_{\mathrm{f}},\end{aligned}$$

where $\mathbb{E}_z[\cdot]$ denotes the expected value with respect to z. It can be shown that, if u_1 and x_{f} are nonzero, the previous equation vanishes only in the limit $\varepsilon \to 0$. This implies that all data-driven expressions in this simple case are biased. When $n > 1$, a quantitative characterization of the bias (and covariance) of the data-driven expressions appears to be difficult, due to the presence of pseudoinverse operations. However, numerical simulations with i.i.d. normally distributed noise (see also Fig. 2) suggest that (i) all data-driven expressions are biased in the case of noisy measurements, (ii) the magnitude of the bias is proportional to the standard deviation σ of the noise for (11) and (18), while it increases rapidly as σ grows and sets to a constant value for (9).

Remark 4 (Data-Driven Expressions With Output Measurements): Consider the system

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where $C \in \mathbb{R}^{p\times n}$, and assume that for each experimental input u_i, $i \in \{1, \ldots, N\}$, we can measure the output of the system at time T, namely, $y_i = Cx_i$. Let $Y = [y_1\ \cdots\ y_N] \in \mathbb{R}^{p\times N}$ be the matrix concatenating all output measurements, and assume that the system is output controllable in T steps. That is, the T-steps output controllability matrix $C_{O,T} = [CB\ CAB\ \cdots\ CA^{T-1}B]$ has full row rank [15]. The minimum-energy input to reach the output $y_{\mathrm{f}} \in \mathbb{R}^p$ in T steps is $u^* = C_{O,T}^\dagger (y_{\mathrm{f}} - CA^T x_0)$.

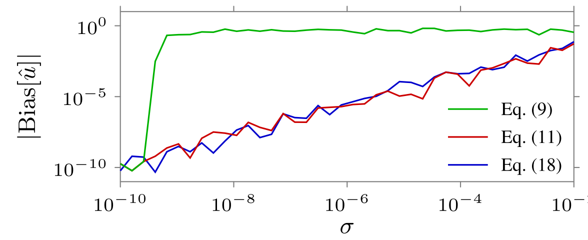


Fig. 2. This figure shows the magnitude of the bias of the data-driven expressions (9), (11), and (18) as a function of the standard deviation of the noise σ. We choose $A = \begin{bmatrix} -0.8 & 0 & 0 \\ 2 & 0.1 & 0 \\ 0.2 & 1 & 0.5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $x_{\mathrm{f}} = \begin{bmatrix} 0.3 \\ 1 \\ 0.5 \end{bmatrix}$, $T = 8$, and $N = 10$. The entries of U and X have been chosen randomly and then corrupted by i.i.d. Gaussian noise with zero mean and standard deviation σ. The bias has been computed as the average over 100 noise realizations.

All results discussed in this letter apply to the case of output control after substituting X and x_{f} with Y and y_{f}, respectively.

III. Numerical Analysis

What remains unclear from the previous analysis is the benefit, if any, in collecting a large number of control experiments. We next show that increasing the number of control experiments can improve the numerical reliability and accuracy of computing minimum-energy control inputs.

In Fig. 3 we compare the numerical performance of the model-based expressions of the minimum-energy controls $u^* = C_T^\dagger x_{\mathrm{f}}$ and $u^* = C_T^{\mathsf{T}} W_T^\dagger x_{\mathrm{f}}$ (Gramian-based), with our data-driven expressions in (9), (11), and (18). In particular, in Fig. 3(a)-(b) we plot the norm of the control inputs and the numerical errors in reaching the final state x_{f}, for all strategies and as a function of the number N of control experiments. Here, we focus on a "worst-case" analysis and choose a small input dimension ($m = 2$), since a large value of m certainly improves the conditioning of all expressions. Fig. 3(a) shows that the norm of the data-driven control inputs (9) and (11) equals its minimum value when $N \geq mT$ (as predicted by Theorems 1 and 2), whereas the norm of the data-driven input (18) converges to its minimum value only asymptotically (as predicted by Theorem 3). Fig. 3(b) shows that, for sufficiently large N, the final state reached by the three data-driven control strategies is almost as close to x_{f} as the one computed via the model-based formula $u^* = C_T^\dagger x_{\mathrm{f}}$, and considerably closer to x_{f} than the state reached by the Gramian-based control input, with expressions (9) and (18) being the most accurate, showing that the computation of the minimum-energy control input via our data-driven expression is as reliable as the computation of the input based on the exact knowledge of the system matrices, and numerically more reliable than the model-based Gramian formula. Instead, in Fig. 3(c)-(d) we plot the norm of the control inputs obtained through the different strategies described above and their corresponding errors in the final state as a function of the system dimension n. As expected, the accuracy of the Gramian-based control input deteriorates rapidly as n increases. Yet, surprisingly, the data-driven expressions of the minimum-energy control inputs remain accurate for systems of considerably larger dimension. Further, the data-driven control (18) yields the smallest error in the final state among the three data-driven

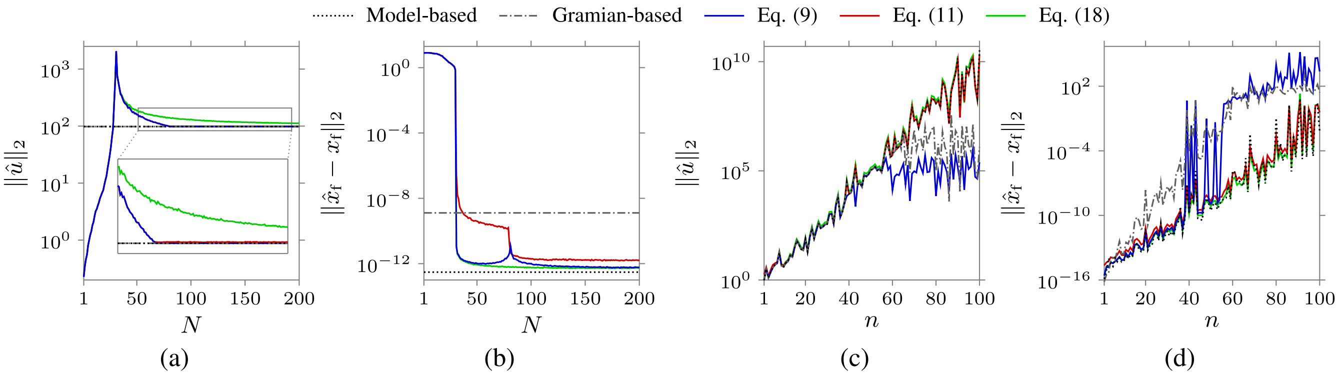


Fig. 3. Panels (a)-(b) show the norm of the control input $\hat{u}$ computed via the model-based formula (5) (dotted line), via inversion of the controllability Gramian (dashed line), and via the data-driven expressions (9), (11), and (18) extended for $x_0 \neq 0$ as in Remark 2 (colored lines), and the corresponding error in the final state, as the number of data N varies. We choose $n = 20$, $m = 2$, and $T = 40$. The matrix A (possibly unstable) has been populated with random i.i.d. normal entries and then normalized by $\sqrt{n}$, the entries of B, x_0 and x_f have been chosen randomly according to a normal distribution. The curves represent the average over 100 experiments with data pairs (x_i, u_i), where u_i has random i.i.d. normal entries, and $x_i = C_T u_i$. Panels (c)-(d) show the norm of the inputs $\hat{u}$ computed as above, and the corresponding errors in the final state, as a function of the system dimension n. We choose $m = 2$, $T = n$, and $N = mT + 20$. The matrices A and B have been generated as above. The curves represent the average over 1000 experiments with data (x_i, u_i) and states x_0, x_f generated as above. All the computations have been carried out using standard built-in `MATLAB 2016b` linear algebra routines.

strategies. This could be due to the simpler form of (18), which requires the computation of only one pseudoinverse, or to the fact that the energy of (18) reaches the minimum value only asymptotically in N. Finally, Fig. 3(c)-(d) show that expression (9) becomes numerically unreliable for smaller values of the system dimension compared to (11) and (18). This is likely because of the additional computations in (9).

IV. Conclusion and Future Work

In this letter, we derive data-driven expressions of open-loop minimum-energy control inputs for linear systems. Leveraging linearity of the dynamics, we show that such optimal controls can be learned from a finite number of control experiments, without knowing or reconstructing the system matrices, and where the control experiments are conducted with non-optimal and arbitrary inputs. We derive three different data-driven expressions of minimum-energy controls: while (11) appears to be the simplest exact data-driven expression, (9) constitutes a radically different and new way of computing minimum-energy controls, and highlights several geometric connections between the minimum-energy solutions and the experimental data, and (18) provides a simple way of computing a family of data-driven, sub-optimal, minimum-energy controls. We further illustrate that our data-driven expressions of the minimum-energy inputs are simpler and numerically more reliable than the classic Gramian-based expression, especially when the dimension of the system increases.

The results of this letter support the intriguing idea of combining model-based control methods with data-driven techniques, showing that this new framework has the potential to considerably increase the reliability and effectiveness of the two parts alone. This letter also creates several directions of future research, including the extension to closed-loop, noisy, and model predictive control problems.

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