Tracking AR(1) Process with limited communication

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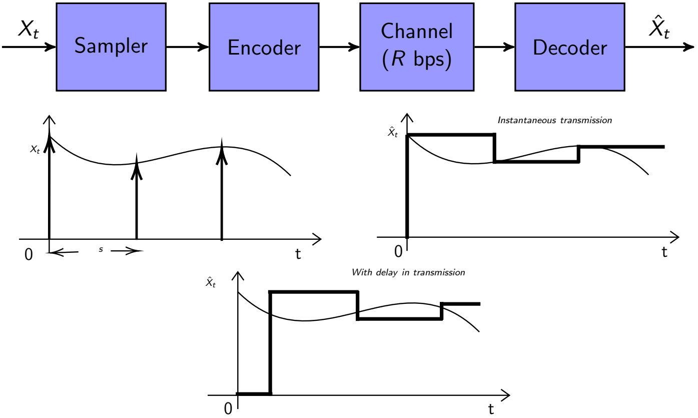
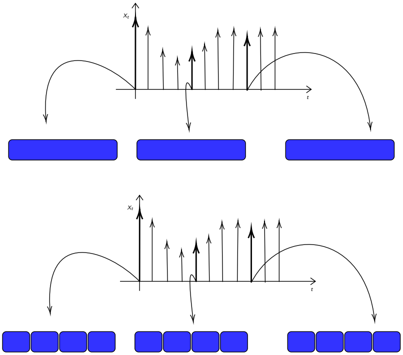
Remote real-time tracking



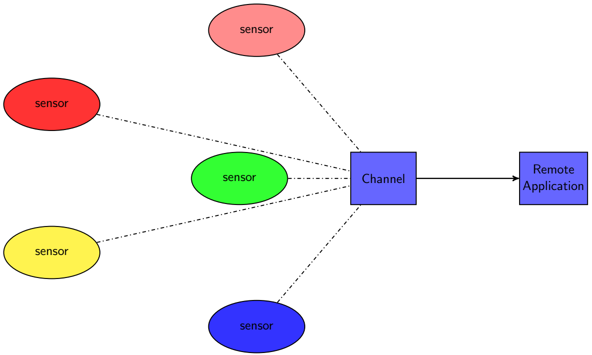
Fast or Precise?

- What is the optimal strategy for real-time tracking of a discrete time process under periodic sampling?
- **Slow and precise** or **Fast but loose**



Application

- Many cyber-physical systems often employ tracking of sensor data in real time
- Examples: sensing, surveillance, real-time control, ...



- Communication is limited by the following constraints:
  - Cost of frequent sampling
  - Limited channel resources

Existing Works

Sequential coding for correlated sources

- Rate-distortion region characterization [**Viswanathan2000TIT**]
- Real-time encoding for Gauss-Markov source [**Khina2017ITW**]

Remote estimation under communication constraints

- Real-time estimation of Wiener process [**Sun2017ISIT**]
- Real-time estimation of AR source [**Chakravorty2017TAC**]

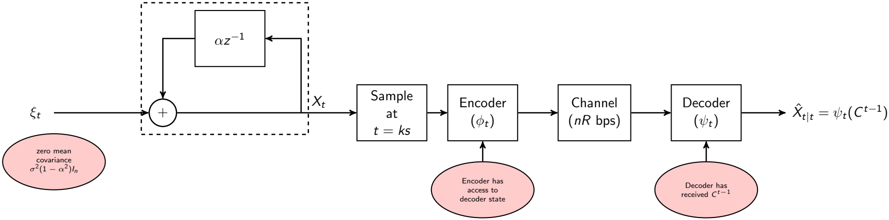
Recursive state estimation algorithms under communication constraints

- Gaussian AR process [**Stavrou2017ITW**]
- Linear system over lossy channel [**Matveev2003TAC**]

Current setting

- Rate-limited channel with unit delay per channel use
- Real-time estimation of AR(1) process

Source Process

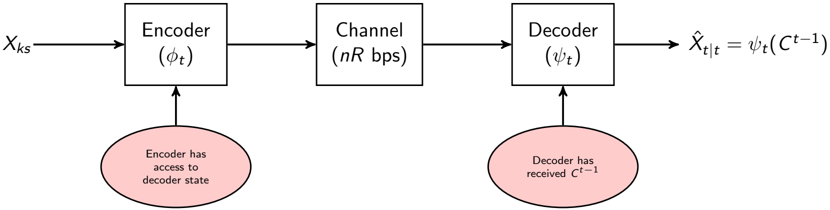


- Innovation process $\xi_t \in \mathbb{R}^n$ is *i.i.d.* and n-dimensional
- Discrete AR(1) n-dimensional source process

$$X_t = \alpha X_{t-1} + \xi_t \quad \text{for all } t \geq 0$$

- Source process X_t is sub-sampled at $1/s$, to obtain samples X_{ks} at $t = ks$
- $\sup_{k \in \mathbb{Z}^+} \frac{1}{n}\sqrt{\mathbb{E}\|X_k\|_2^4}$ is bounded

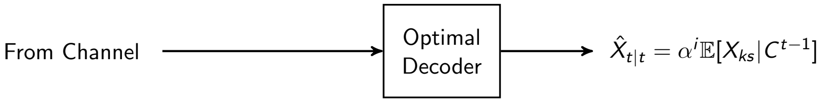
Communication Setting



- **Encoder**: $\phi_t : \mathcal{X}^{k+1} \to \{0,1\}^{nRs}$ at $t = ks$
- **Channel**: Error free, limited capacity causes delayed transmission
- **Decoder**: $\psi_t : \{0,1\}^{nR(t-1)} \to \mathcal{X}$ at $t = ks$
- **Performance metric**:

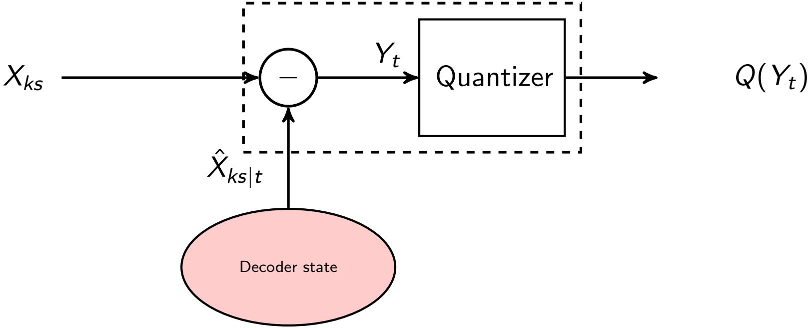
$$D_t(\phi, \psi, X) \triangleq \frac{1}{n}\mathbb{E}\|X_t - \hat{X}_{t|t}\|_2^2.$$

Optimal Decoder Structure



- Decoder at time $t = ks + i$ for $i \in \{1, \ldots, s\}$
- For the mean squared error, estimate conditional mean
- Utilize the latest information to refine the last sample X_{ks}

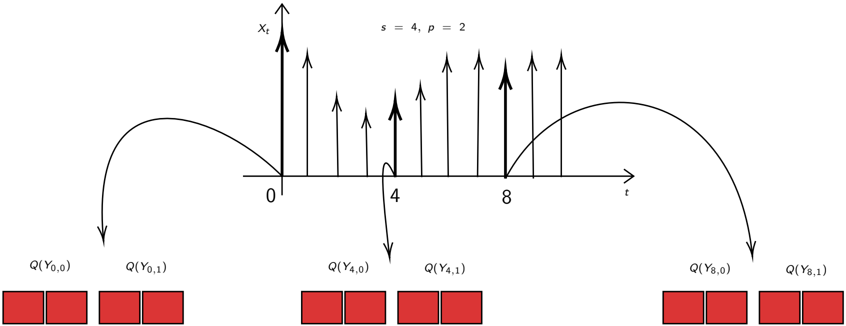
Encoder Structure



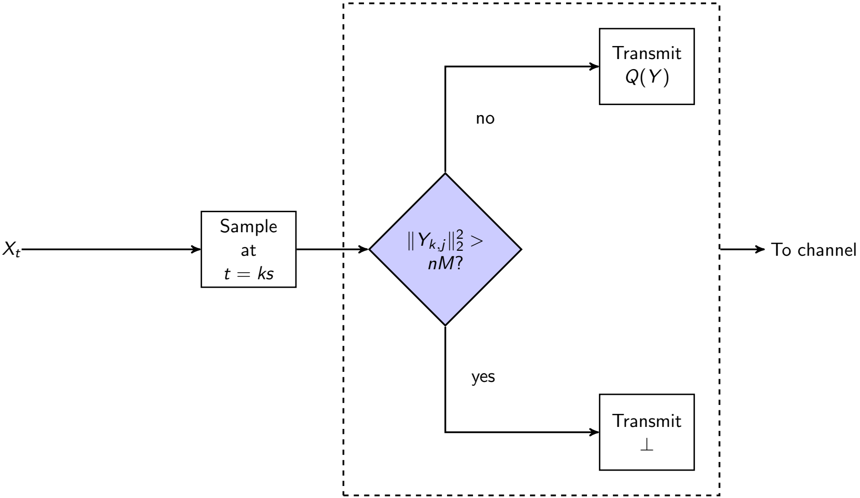
- Find the error in the decoder estimate of the last sample
- Transmit the quantized error

Periodic Successive Update Scheme

- At $t = ks + jp$, $j \in [0, s/p - 1]$, encode $Y_{k,j} = X_{ks} - \hat{X}_{ks|ks+jp}$.



Encoder at time $t = ks + jp$



(θ, ε)-quantizer

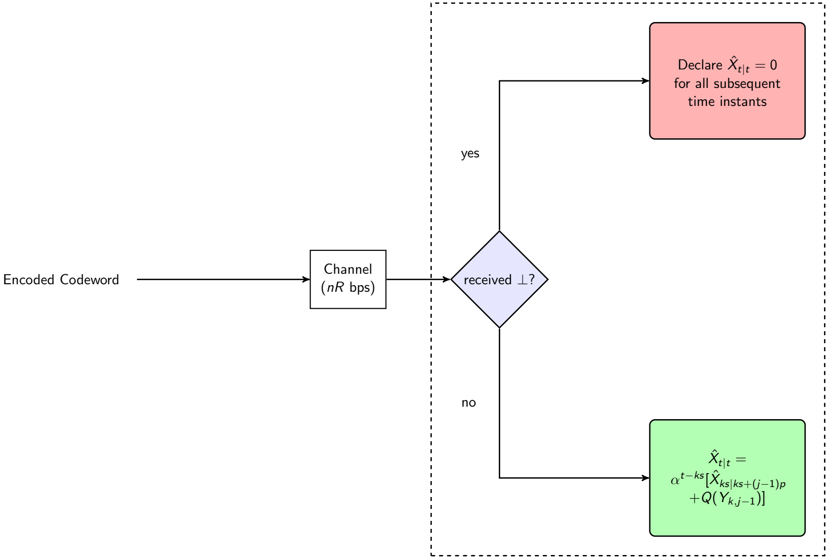
Definition

Fix $0 < M < \infty$. A quantizer $Q : \mathbb{R}^n \to \{0, 1\}^{nR}$ constitutes an nR bit (θ, ε)-*quantizer* if for every vector $y \in \mathbb{R}^n$ such that $\frac{1}{n}\|y\|_2 \leq M$, we have

$$\mathbb{E}\|y - Q(y)\|_2^2 \leq \|y\|_2^2 \theta(R) + n\varepsilon^2.$$

for $0 \leq \theta \leq 1$ and $0 \leq \varepsilon$.

Decoder at time $t = ks + jp + i$



Performance of Periodic Successive Update Scheme

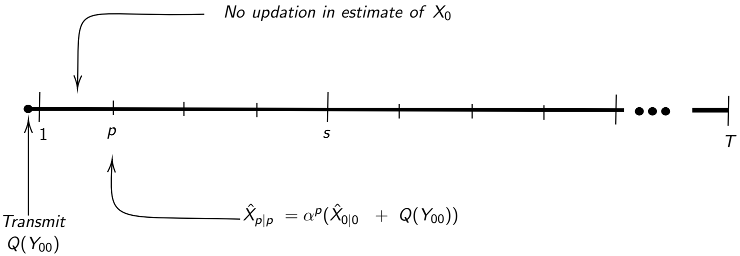
Lemma

For $t = ks + jp + i$, the p-SU scheme employing a nRp bit (θ, ϵ) quantizer satisfies

$$D_t(\phi_p, \psi_p, X) \leq \alpha^{2(t-ks)}\theta(Rp)^j D_{ks}(\phi_p, \psi_p, X) + \sigma^2(1 - \alpha^{2(t-ks)}) + f(\epsilon, \beta).$$

β : Upperbound on the probability of encoder failure

Proof Idea



- $X_p = \alpha^p X_0 + \sum_{u=1}^{p} \alpha^{p-u}\xi_u$
- When encoding is successful, $\hat{X}_{p|p} = \alpha^p \hat{X}_{0|p}$,

$$\begin{aligned} D_p &= \alpha^{2p}\frac{1}{n}\mathbb{E}\|X_0 - \hat{X}_{0|0} - Q(X_0 - \hat{X}_{0|0})\|_2^2 + \sigma^2(1-\alpha^{2p}) \\ &\leq \alpha^{2p}\theta\frac{1}{n}\mathbb{E}\|X_0 - \hat{X}_{0|0}\|_2^2 + \epsilon^2 + \sigma^2(1-\alpha^{2p}) \end{aligned}$$

- Else, use Cauchy-Schwartz Inequality

Performance of Periodic Successive Update Scheme

Lemma

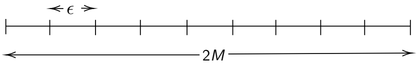
For a fixed time horizon T, periodic successive update scheme with a (θ, ϵ) quantizer gives

$$\frac{1}{T}\sum_{t=0}^{T} D_t(\phi_p, \psi_p, X) \leq \sigma^2 \left[1 - \frac{g(s)\,\alpha^{2p}}{1-\alpha^{2p}\,\theta(Rp)}\left(1 - \frac{\varepsilon^2}{\sigma^2} - \theta(Rp)\right)\right]$$

for a very low probability of encoder failure and $g(s) \triangleq \frac{1-\alpha^{2s}}{s(1-\alpha^2)}$.

Example: 1 Uniform Quantizer

$R = \log\lceil 2M/\epsilon \rceil$



- Say we quantize y, $\|y\|_2^2 \leq \sqrt{n}M$
- The quantizer parameters : $\theta = 0$, $\epsilon^2 = nM^2 2^{-2R}$
- Optimal p is 1

Example:2 Average Distortion Upper Bound for Gain-Shape Quantizer

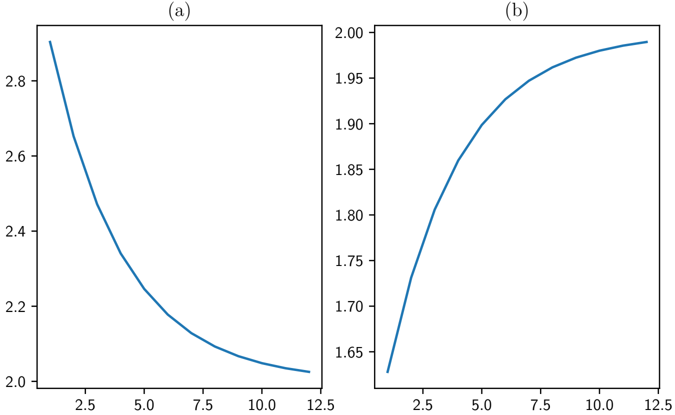
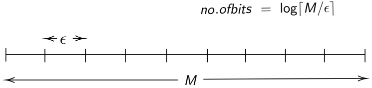


Figure: (a) gives a case where p =s is the best and in (b) p =1 minimizes the bound

A quantizer design

Norm Quantizer : Quantizes the norm $B = \|y\|_2/\sqrt{n}$ into $\hat{B}$ such that $|B - \hat{B}| \leq \varepsilon$.



Angle Quantizer[1] : A random codebook $\mathcal{C}$ consisting of 2^{nR} independent vectors distributed uniformly over the unit sphere $\mathbb{S}$ in $\mathbb{R}^n$.

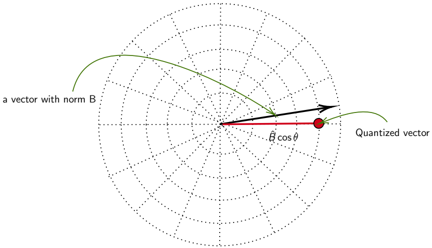
For any unit vector $y \in \mathbb{R}^n$,the quantizer gives $Q(y) = \sqrt{n}\cos\theta \cdot \arg\max_{y' \in \mathcal{C}} \langle y, y' \rangle$.

θ chosen to guarantee that there is one codeword y' such that, $\langle y, y' \rangle > \cos\theta$ for all y.

[1] Amos Lapidoth. "On the role of mismatch in rate distortion theory". In: *IEEE Trans. Inf. Theory* 43.1 (1997), pp. 38–47.

Performance of the quantizer

For any $y \in \mathbb{R}^n$,the quantizer gives
$Q(y) = \sqrt{n}\,\hat{B}\,\cos\theta \cdot \arg\max_{y' \in \mathcal{C}} \langle y, y' \rangle$.



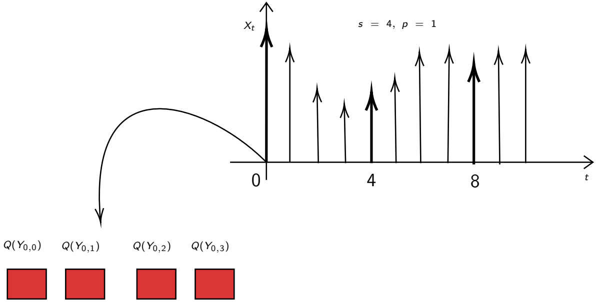
Lemma

Consider a vector $y \in \mathbb{R}^n$ with $\|y\|_2^2 = nB^2$. Suppose that $B \le M$ and let $|B - \hat{B}| \le \varepsilon$. Then,

$$\frac{1}{n}\|y - Q(y)\|_2^2 \le 2^{-2(R')}B^2 + \varepsilon^2.$$

Special Case: Successive Update scheme

- Fast and Loose
- Set p = 1



Performance of the scheme

Lemma

Let $t = ks + i$, for $i \in [1, s]$, for n sufficiently large, the successive update scheme used with a (θ, ϵ) quantizer realisation with $\theta(R) = 2^{-R}$ satisfies

$$D_t(\phi, \psi, X) \leq \alpha^{2i} 2^{-2Ri} D_{ks}(\phi, \psi, X) + \sigma^2(1 - \alpha^{2i}) + f_n$$

where $f_n \to 0$ for large n.

Optimum min-max tracking accuracy

Definition

We define the accuracy,

$$\delta^T(\phi, \psi, \mathbb{X}_n) = 1 - \frac{\frac{1}{T}\sum_{t=0}^{T-1} D_t(\phi, \psi, X)}{\sigma^2}$$

Then, optimum asymptotic maxmin tracking accuracy,

$$\delta^*(R, s, \mathbb{X}) = \lim_{T\to\infty} \lim_{n\to\infty} \left[\sup_{(\phi,\psi)} \inf_{X\in\mathbb{X}_n} \delta^T(\phi, \psi, \mathbb{X}_n) \right].$$

Main Result

Theorem (Lower bound for maxmin tracking accuracy: The achievability)

For $R > 0$ and $s \in \mathbb{N}$, the asymptotic minmax tracking accuracy is bounded below as

$$\delta^*(R, s, \mathbb{X}) \geq \delta_0(R) g(s).$$

for $\delta_0(R) \triangleq \frac{\alpha^2(1-2^{-2R})}{(1-\alpha^2 2^{-2R})}$ *and* $g(s) \triangleq \frac{(1-\alpha^{2s})}{s(1-\alpha^2)}$ *for all* $s > 0$.

This bound is achieved using successive update scheme for $p = 1$ and the given realisation of (θ, ϵ) quantizer.

Theorem (Upper bound for maxmin tracking accuracy: The converse)

For $R > 0$ and $s \in \mathbb{N}$, the asymptotic minmax tracking accuracy is bounded above as

$$\delta^*(R, s, \mathbb{X}) \leq \delta_0(R) g(s).$$

The upper bound is obtained by considering the Gauss-Markov Processes.

Conclusion

- We provide an information theoretic upper bound for maxmin tracking accuracy for a fixed rate and sampling frequency.
- It is shown that for a fixed rate, high dimensional setting, the strategy of being *fast but loose* achieves this bound.
- We outline the performance requirements of the quantizer needed for achieving the optimal performance.
- For non-asymptotic regime our studies show that the optimal strategy might differ.

References I

Amos Lapidoth. "On the role of mismatch in rate distortion theory". In: *IEEE Trans. Inf. Theory* 43.1 (1997), pp. 38–47.