Double-Auction Mechanisms for Resource Trading Markets

K. P. Naveen and Rajesh Sundaresan

***Abstract*—We consider a double-auction mechanism, which was recently proposed in the context of a mobile data-offloading market. It is also applicable in a network slicing market. Network operators (users) derive benefit from offloading their traffic to third party WiFi or femtocell networks (link-suppliers). Link-suppliers experience costs for the additional capacity that they provide. Users and link-suppliers (collectively referred to as agents) have their pay-offs and cost functions as private knowledge. A system-designer decomposes the problem into a network problem (with surrogate pay-offs and surrogate cost functions) and agent problems (one per agent). The surrogate pay-offs and cost functions are modulated by the agents' bids. Agents' payoffs and costs are then determined by the allocations and prices set by the system designer. Under this design, so long as the agents do not anticipate the effect of their actions, a competitive equilibrium exists as a solution to the network and agent problems, and this equilibrium optimizes the system utility. However, this design fails when the agents are strategic (price-anticipating). The presence of strategic supplying agents drives the system to an undesirable equilibrium with zero participation resulting in an efficiency loss of 100%. This is in stark contrast to the setting when link-suppliers are not strategic: the efficiency loss is at most 34% when the users alone are strategic. The paper then proposes a Stackelberg game modification with asymmetric information structures for suppliers and users in order to alleviate the efficiency loss problem. The system designer first announces the allocation and payment functions. He then invites the supplying agents to announce their bids, following which the users are invited to respond to the suppliers' bids. The resulting Stackelberg games' efficiency losses can be characterized in terms of the suppliers' cost functions when the user pay-off functions are linear. Specifically, when the link-supplier's cost function is quadratic, the worst case efficiency loss is 25%. Further, the loss in efficiency improves for polynomial cost functions of higher degree.**

***Index Terms*—Network utility maximization, double-auction, KKT conditions, Nash equilibrium, Stackelberg equilibrium.**

I. Introduction

We consider double auction mechanisms motivated by two examples – mobile data offloading and network slicing-based virtualization.

Mobile data offloading is an effective way to manage growth in mobile-data traffic. Traffic meant for the macrocellular network can be offloaded to already installed third-party Wi-Fi or femtocell networks. This provides an alternative means of network expansion. Wi-Fi access-point operators and femtocell network operators will however expect compensation for allowing macrocellular network traffic through their access points. Technological, security, and preliminary economic studies for secure and seamless offloading have been discussed in [2]–[5].

Network slicing [6] is a virtualization technique that allows many logical networks to run atop shared physical networks. It allows physical mobile network operators to partition their network resources and offer them to different users or tenants (IoT streams, mobile broadband streams, etc.) in return for suitable compensation. It enables network operators to focus on their core strength of delivering high-quality network experiences while the tenants or virtual network operators can focus more on business, billing, and branding relations.

In the rest of the paper, we discuss double auction mechanisms in the context of mobile data offloading. But the mapping to the context of network slicing will be obvious.

In recent work, Iosifidis et al. [7] proposed a double-auction mechanism where a set of mobile network operators (buyers or *users* in this work) compete for resources from access-point operators (sellers or *links* in this work). The pay-offs of the users and costs of the links are private information to the respective parties. The mechanism works as follows. A network manager collects how much each network operator is willing to pay each access-point operator, scalar signals on the costs at each access point, and then determines how much traffic should be offloaded to each access point and how much each agent will pay or get. The mobile network operators and the access-point operators then comply. This is a scenario with an asymmetric information structure where (a) the broker is not aware of the actual needs and costs of network and access-point operators, (b) each operator is aware only of his own needs or costs, and (c) all agents are *price-taking* (made precise in the next section). Following Kelly et al. [8], Iosifidis et al. [7] showed that a tâtonnement procedure converges to the system optimal operating point.

Iosifidis et al. [7, p.1635] point out that designing incentive compatible mechanisms for double-auctions which are weakly budget balanced (the broker should not end up subsidizing the mechanism) is 'notoriously hard' and has been done only in certain simplified settings (McAfee auction [9]) or can be computationally intensive. So [7] took a network utility maximization approach and left the analysis of the price-anticipating scenario open [7, Sec VII, p.1646].

Our contributions in this paper are as follows.

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1) We first re-derive the result on efficient allocation when the agents are price-taking, mainly to set up the notation for the next three results.
2) We then analyze the price-anticipating scenario along the lines of Johari et al. [10]. When agents are price-anticipating, they recognize the effect of their bids on the allocation. The appropriate equilibrium notion is a Nash equilibrium. The situation in Johari et al. [10], when mapped to the current offloading setting, would be one where the access-point operators are not strategic. The efficiency loss due to price-anticipating mobile offloading agents is then at most 34%. However, when the access-point agents (suppliers) are also strategic and price-anticipating, the equilibrium is one where the offloading agents prefer not to offload any traffic. The efficiency loss is then 100%. The main message is that the earlier proposed double-auction mechanism of [7] works when agents are price-taking, but fails in the more real situation when agents are price-anticipating. One must then look for alternative double-auction mechanisms.
3) We then propose a modified mechanism where the supplying agent bids first and the users bid in response. To show that the situation is now improved, we characterize the new efficiency loss in terms of the supplier's cost function, when the user pay-off functions are linear. For instance, for the quadratic link-cost function, the worst-case efficiency loss (with the worst-case taken over linear user pay-off functions) is at most 25%.
4) We extend all of the above results to the setting with multiple links.

From an implementation theory perspective, the Iosifidis et al. [7] mechanism in the price-taking scenario implements the social welfare maximization rule under the competitive equilibrium solution concept with the minimal message dimension of 1 (scalar signals). The above implementation ignores strategic behavior of individual agents. It is not possible to enforce such mechanisms in general because individual preferences may diverge from social welfare maximization. This is the price-anticipating scenario. It is anticipated that if we do not enlarge the signal space dimension there may be no mechanism, let alone the Iosifidis et al. mechanism, that can implement the social welfare maximization rule, under now the Nash equilibrium solution concept. This is why the price-anticipating scenario with non-strategic link suppliers suffered from an efficiency loss. What is surprising in our current setting is the dramatic increase in efficiency loss from at most 34% (Johari et al. [10]) to 100% (contribution (2) of this paper). What is promising from our study is that this efficiency loss can be mitigated by structuring the interaction, by making the link player lead the interaction (contribution (3) of this paper). The solution concept is that of a Stackelberg equilibrium. Efficiency loss drops down to a value that depends on the supplier's cost function and is at most 25% for quadratic costs and linear user pay-offs. This of course raises the question of what is the minimal signalling dimension in the price-anticipating scenario that implements the social welfare maximization rule in the Nash equilibrium solution concept. This a very interesting question that is beyond the scope of this work. Our proposed scheme, which structures the interactions by asking the supplier to lead, reduces efficiency loss. It would be of utmost interest if this structuring also reduces the minimum signalling dimension for social welfare maximization in the Stackelberg equilibrium solution concept. We refer the reader to [11] for an excellent discussion on the implementation theory perspective.

The paper is organized into two parts. In Part I we study a setting with a single link-supplier. Specifically, in Section II, we discuss the system model and problem definition. In Section III, we discuss the price-taking scenario for the single-link case. In Section IV, we analyze the price-anticipating scenario. As a positive result, in Section V, we discuss our proposed mechanism and characterize the worst-case efficiency loss for linear user pay-offs in terms of the single supplying agent's cost function. In Part II (Sections VI to IX) we generalize the above results to the setting with multiple link-suppliers. To focus on the flow of key ideas, we have moved all the proofs to the Appendix. The paper concludes with some remarks in Section X.

Part I: Single Link

II. System Model and Problem Definition

Consider a scenario where M *users* intend to share the bandwidth of a (single) *link* of capacity $C > 0$ owned by a *link-supplier*. In the context of mobile-data offloading [7], users and link-supplier correspond to mobile-network operators and the single access-point operator (e.g., Wi-Fi, femtocell), respectively. The mobile-network operators want to buy a share of the limited bandwidth resource available at the access point to offload their macrocellular traffic, while the access point operator is interested in maximizing his profit. In the double auction terminology [9], users are synonymous to buyers bidding for a share of a resource while the link-supplier is the seller. We refer to the users and the link-supplier collectively as *agents*. The social planner, the entity that designs the mechanism (i.e., sets up the rules for information transfer, allocation, and payments) is referred to as the *network-manager*,

Let x_m denote the rate requested by user $m = 1, 2, \cdots, M$, and let y_m be the rate the link-supplier is willing to allocate to user m. Thus, $\mathbf{x} = (x_1, x_2, \cdots, x_M)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_M)$ represent the *rate-request* and *rate-allocation* vectors, respectively. Let $y = \sum_m y_m$ denote the aggregate-rate allocated by the link-supplier to all users. For user m, the benefit of acquiring a rate of x_m is represented by a pay-off function $U_m(x_m)$; we assume that U_m, $m = 1, 2, \cdots, M$, are concave, strictly increasing and continuously differentiable with finite $U'_m(0)$. Similarly, the cost incurred by the link-supplier for accepting to serve an aggregate rate of y is given by $V(y)$, where V is strictly convex, strictly increasing and continuously differentiable. Thus, the system optimal solution is the solution to the optimization problem:

SYSTEM

$$\text{Maximize: } \sum_m U_m(x_m) - V\left(\sum_m y_m\right) \tag{1a}$$

$$\text{Subject to: } \sum_m y_m \leq C \tag{1b}$$

$$x_m \leq y_m \ \forall m \tag{1c}$$

$$x_m \geq 0, y_m \geq 0 \ \forall m. \tag{1d}$$

Continuity of the objective function and compactness of the constraint set imply that an optimal solution $\mathbf{x}^s = (x_1^s, x_2^s, \cdots, x_M^s)$ and $\mathbf{y}^s = (y_1^s, y_2^s, \cdots, y_M^s)$ exists. Further, if U_m are strictly concave then (since V is strictly convex) the solution is unique. Since U_m are strictly increasing in x_m, an optimal solution must satisfy $\mathbf{x}^s = \mathbf{y}^s$. Thus, at optimality, the rate-requests (demand) and the rate-allocations (supply) are matched although the capacity C may not be fully utilized.

A network-manager, however, cannot solve the formulation in (1) without the knowledge of user pay-offs and the link-cost function. Hence, consider the following mechanism proposed by Iosifidis et al. in [7] for rate allocation. Each user m submits a *bid* $p_m \geq 0$ that denotes the amount he is willing to pay, while the link-supplier communicates signals β_m $(m = 1, 2, \cdots, M)$ that implicitly indicate the amounts of bandwidth that he is willing to provide; we refer to $\mathbf{p} := (p_1, p_2, \cdots, p_M)$ and $\boldsymbol{\beta} := (\beta_1, \beta_2, \cdots, \beta_M)$ as the *bids* submitted by the users and the link-supplier, respectively.

The network-manager is responsible for fixing the *prices* μ_m $(m = 1, 2, \cdots, M)$ and λ that determines the rate allocation. The prices $\boldsymbol{\mu} := (\mu_1, \mu_2, \cdots, \mu_M)$ and λ are supposed to be the optimal dual variables of the following network problem proposed by Iosifidis et al. in [7]:

NETWORK

$$\text{Maximize: } \sum_m p_m \log(x_m) - \sum_m \frac{y_m^2}{2\beta_m} \tag{2a}$$

$$\text{Subject to: } \sum_m y_m \leq C \tag{2b}$$

$$x_m \leq y_m \ \forall m \tag{2c}$$

$$x_m \geq 0, y_m \geq 0 \ \forall m. \tag{2d}$$

In the NETWORK problem above we choose to use $\boldsymbol{\beta}$ instead of a related $\boldsymbol{\alpha}$ that was used in the original formulation by Iosifidis et al. in [7]; the quantities $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related by $\beta_m = 1/\alpha_m \ \forall m$. Then each β_m is $\mathbb{R}_+$-valued, with values on the positive real line, while each α_m is in general $\mathbb{R}_+ \cup \{+\infty\}$-valued. Moreover, the signals in $\boldsymbol{\beta}$ are directly proportional to the amount of bandwidth the link-supplier is willing to share. For instance, a lower value of β_m implies that the bandwidth shared by the link-supplier with user m is low, and vice versa. In particular, $\beta_m = 0$ implies that the link-supplier is unwilling to share any bandwidth with user m. This will be useful while interpreting the Nash equilibrium bid-vectors (Theorem 2).

The above NETWORK problem is identical to the SYSTEM problem but with the true pay-off and cost functions replaced by *surrogate* pay-off and cost functions. In the following, we first review the case when the users and the link-supplier are *price-taking*. This means agents assume prices are given and do not anticipate the effect of their bids on the prices set by the network-manager. See Definition 1 below of a competitive equilibrium. We then proceed to study the more-involved *price-anticipating* scenario. Here the agents recognize that the effective price is based on their bids, anticipate the resulting allocation, payment, and therefore their pay-off, and act accordingly. The resulting pay-off functions are new functions of the bids; see Definition 2. Our methodology in Sections III and IV is similar to Johari et al. [10], but the outcome in the price-anticipating scenario is dramatically negative due to the presence of the strategic link-supplier, as we will soon see. We then propose a remedy via a Stackelberg framework where the link-supplier is a lead player and the users are followers.

III. Price-taking scenario

The sequence of exchanges (between the network-manager and the agents) in the price-taking scenario is as shown in the box describing the price-taking mechanism (PTM) below.

PRICE-TAKING MECHANISM (PTM)

1) The network manager announces to the agents how the allocation will be done and the payments will be fixed, as a function of prices and agents' bids.
2) The network-manager then initiates the bidding process by fixing the prices $(\boldsymbol{\mu}, \lambda)$.
3) The agents accept the prices and respond by announcing their respective bids, $\mathbf{p}$ and $\boldsymbol{\beta}$.
4) The network-manager allocates a rate of $x_m = p_m/\mu_m$ to user m and receives a payment of p_m. Simultaneously, the link-supplier is asked to allocate a rate of $y_m = \beta_m(\mu_m - \lambda)$ to user m; the total payment made to the link-supplier is $\sum_m \beta_m(\mu_m - \lambda)^2$.

The prices set by the network-manager are $(\boldsymbol{\mu}, \lambda)$. The pay-off to user m, for bidding p_m, is given by

$$P_m\Big(p_m; \mu_m\Big) = U_m\left(\frac{p_m}{\mu_m}\right) - p_m. \tag{3}$$

Similarly, the pay-off to the link-supplier is given by

$$P_L\Big(\boldsymbol{\beta}; (\boldsymbol{\mu}, \lambda)\Big) = -V\left(\sum_m \beta_m\Big(\mu_m - \lambda\Big)\right) + \sum_m \beta_m\Big(\mu_m - \lambda\Big)^2. \tag{4}$$

Using the above pay-off functions we characterize the solution as a *competitive equilibrium* which is defined as follows (unless mentioned otherwise, we assume that the agents' bids and the link-supplier's prices are non-negative, i.e., $p_m, \beta_m, \mu_m, \lambda \geq 0 \ \forall m$; also, we use $\mathbf{0}$ to denote the vector of all-zeros of appropriate length):

Definition 1 (Competitive Equilibrium [10], [12]): We say that $(\mathbf{p}, \boldsymbol{\beta}, \lambda, \boldsymbol{\mu})$ constitutes a competitive equilibrium if the following conditions hold:

(C1) $P_m(p_m; \mu_m) \geq P_m(\overline{p}_m; \mu_m) \ \forall \overline{p}_m \geq 0, \forall m$

(C2) $P_L\Big(\boldsymbol{\beta}; (\boldsymbol{\mu}, \lambda)\Big) \geq P_L\Big(\overline{\boldsymbol{\beta}}; (\boldsymbol{\mu}, \lambda)\Big) \ \forall \overline{\boldsymbol{\beta}} \geq \mathbf{0}$

(C3) Define $\mathcal{M} = \left\{m : \mu_m \neq \lambda\right\}$ and

$$\widehat{C} = \sqrt{\left(\sum_m p_m\right)\left(\sum_{m\in\mathcal{M}} \beta_m\right)}. \qquad (5)$$

Then, the following should hold:

(C3-a) For all m,

$$\frac{p_m}{\mu_m} = \beta_m\Big(\mu_m - \lambda\Big); \qquad (6)$$

(C3-b) For all $m \in \mathcal{M}$, the equality $\mu_m = \mu$ holds, where

$$\mu = \sum_i p_i \Big/ \min\left\{C, \widehat{C}\right\}; \qquad (7)$$

(C3-c) Furthermore,

$$\lambda = \min\left\{0, \left(1 - \left(\frac{C}{\widehat{C}}\right)^2\right)\frac{\sum_i p_i}{C}\right\}. \qquad (8)$$

□

In the above definition, condition (C1) implies that the users do not benefit by deviating from their equilibrium bids p_m, when the prices $(\lambda, \boldsymbol{\mu})$ set by the network-manager are fixed. Similarly, (C2) implies that the link-supplier has no benefit in deviating from the equilibrium bid-vector $\boldsymbol{\beta}$. Although (C1) and (C2) result in the optimality of the users' and the link-supplier's problem of maximizing their respective pay-offs, these conditions by themselves do not guarantee system-optimal performance. The conditions in (C3) (essentially derived from the optimality conditions for NETWORK) are crucial to guarantee that the prices $(\lambda, \boldsymbol{\mu})$ set by the network-manager are dual optimal for SYSTEM. Condition (C3) along with (C1) and (C2) can then be used to show the optimality of a competitive equilibrium. We summarize this result in the following theorem; in particular, we first prove the existence of a competitive equilibrium, and then derive its optimality property. This theorem is essentially an extension of the result due to Kelly [13] and Kelly et al. [8] (see also [10] and [12]). The main difference that warrants an extension is the presence of the link-supplier as a strategic agent.

Theorem 1: When the agents are price-taking, there exists a competitive equilibrium, i.e., there exist vectors $(\mathbf{p}, \boldsymbol{\beta}, \lambda, \boldsymbol{\mu})$ satisfying (C1), (C2) and (C3). Moreover, given a competitive equilibrium $(\mathbf{p}, \boldsymbol{\beta}, \lambda, \boldsymbol{\mu})$, the rate vectors $\mathbf{x}$ and $\mathbf{y}$ defined as $x_m = p_m/\mu_m$ and $y_m = \beta_m(\mu_m - \lambda)$ $(\forall m)$ are optimal for the problem SYSTEM in (1).

Proof: The result can be gleaned from the results in [7] though it is not explicitly stated. Our proof of Theorem 1 is a direct one that does not rely on any learning dynamics. Instead, it is based on Lagrangian techniques. Details are available in Appendix A. ■

IV. Price-anticipating scenario

In contrast to the price-taking scenario, agents initiate the bidding process in the price-anticipating scenario. Specifically, the sequence of exchanges is as given below.

PRICE-ANTICIPATING MECHANISM (PAM)

1) The network manager first announces to the agents how the allocation will be done and the payments will be fixed, as a function of prices and agents' bids.
2) Agents then initiate the bidding process by *simultaneously* announcing their bids, denoted $\mathbf{p}$ and $\boldsymbol{\beta}$.
3) The network-manager sets prices $(\boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}), \lambda(\mathbf{p}, \boldsymbol{\beta}))$ where we have set $\boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}) = (\mu_1(\mathbf{p}, \boldsymbol{\beta}), \cdots, \mu_M(\mathbf{p}, \boldsymbol{\beta}))$. Note that the above prices are dual optimal for the NETWORK problem in (2).
4) The payments and the allocated rates are exactly as in the price-taking mechanism, but with $(\boldsymbol{\mu}, \lambda)$ replaced by $(\boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}), \lambda(\mathbf{p}, \boldsymbol{\beta}))$.

In the following lemma we report the expression for the prices $(\lambda(\mathbf{p}, \boldsymbol{\beta}), \boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}))$.

Lemma 1: Given any vector $(\mathbf{p}, \boldsymbol{\beta})$ of users' and link-supplier's bids, the prices $(\lambda(\mathbf{p}, \boldsymbol{\beta}), \boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}))$ set by the network-manager are given by

$$\lambda(\mathbf{p}, \boldsymbol{\beta}) \quad = \quad \begin{cases} 0 & \text{if } \sum_i \sqrt{p_i \beta_i} \leq C \\ f^{-1}_{\mathbf{p},\boldsymbol{\beta}}(C) & \text{otherwise,} \end{cases} \qquad (9)$$

where $f^{-1}_{\mathbf{p},\boldsymbol{\beta}}$ is the inverse of $f_{\mathbf{p},\boldsymbol{\beta}}$ defined as

$$f_{\mathbf{p},\boldsymbol{\beta}}(t) \quad = \quad \sum_i \left(\frac{2p_i}{t + \sqrt{t^2 + 4\frac{p_i}{\beta_i}}}\right), \qquad (10)$$

and for $m = 1, 2, \cdots, M$

$$\mu_m(\mathbf{p}, \boldsymbol{\beta}) \quad = \quad \frac{\lambda(\mathbf{p}, \boldsymbol{\beta}) + \sqrt{\lambda(\mathbf{p}, \boldsymbol{\beta})^2 + 4\frac{p_m}{\beta_m}}}{2}. \qquad (11)$$

Proof: See Appendix B-A. ■

Continuing with the discussion, using the above prices in (3), the pay-offs to the users in the price-anticipating scenario can be expressed as follows for $m = 1, 2, \cdots, M$ (for simplicity, we use $\lambda := \lambda(\mathbf{p}, \boldsymbol{\beta})$):

$$\begin{aligned} Q_m(p_m, \mathbf{p}_{-m}, \boldsymbol{\beta}) &= U_m\left(\frac{p_m}{\mu_m(\mathbf{p}, \boldsymbol{\beta})}\right) - p_m \\ &= \begin{cases} U_m\left(\sqrt{p_m \beta_m}\right) - p_m & \text{if } \sum_i \sqrt{p_i \beta_i} \leq C \\ U_m\left(\frac{2p_m}{\lambda + \sqrt{\lambda^2 + 4\frac{p_m}{\beta_m}}}\right) - p_m & \text{otherwise,} \end{cases} \end{aligned} \qquad (12)$$

where $\mathbf{p}_{-m} = (p_1, \cdots, p_{m-1}, p_{m+1}, \cdots, p_M)$ denotes the bids of all users other than m, while $\boldsymbol{\beta}$ is the bid submitted by the link-supplier. Similarly, for the link-supplier we have

$$\begin{aligned} &Q_L(\boldsymbol{\beta}, \mathbf{p}) \\ &= \begin{cases} -V\left(\sum_m \sqrt{p_m \beta_m}\right) + \sum_m p_m & \text{if } \sum_i \sqrt{p_i \beta_i} \leq C \\ -V(C) + \sum_m \frac{1}{\beta_m}\left(\frac{2p_m}{\lambda + \sqrt{\lambda^2 + 4\frac{p_m}{\beta_m}}}\right)^2 & \text{otherwise.} \end{cases} \end{aligned} \qquad (13)$$

The quantity $V(C)$ in the above expression is due to complementary slackness conditions which imply

$$\sum_m \frac{p_m}{\mu_m(\mathbf{p},\boldsymbol{\beta})} = \sum_m y_m = C \text{ whenever } \lambda > 0.$$

The users and the link-supplier recognize that their bids affect the prices and the allocation. Acting as rational and strategic agents, they now anticipate these prices. The appropriate notion of an equilibrium in this context is the following.

Definition 2 (Nash Equilibrium): A bid vector $(\mathbf{p},\boldsymbol{\beta})$ is a Nash equilibrium if, for all $m = 1, 2, \cdots, M$, we have

$$\begin{aligned} Q_m(p_m, \mathbf{p}_{-m}, \boldsymbol{\beta}) &\geq Q_m(\overline{p}_m, \mathbf{p}_{-m}, \boldsymbol{\beta}) \ \forall \overline{p}_m \geq 0 \\ Q_L(\boldsymbol{\beta}, \mathbf{p}) &\geq Q_L(\overline{\boldsymbol{\beta}}, \mathbf{p}) \ \forall \overline{\boldsymbol{\beta}} \geq \mathbf{0}. \end{aligned}$$

□

When $\sum_i \sqrt{p_i \beta_i} < C$, the link is not fully utilized. In this case the Lagrange multiplier $\lambda = \lambda(\mathbf{p},\boldsymbol{\beta}) = 0$. Examination of (12) and (13) indicates that the payments made by the users are all passed on to the link-supplier. This may be interpreted as follows: for a given set of payments, the link-supplier bids are such that the link is viewed as a costly resource and the network-manager passes on all his revenue to the link-supplier. The link-supplier is thus assured of this revenue even if his link is not fully utilized. If, on the other hand, the link-supplier's bids are such that $\sum_i \sqrt{p_i \beta_i} > C$, then $\lambda > 0$, and it is clear from (13) that not all the collected revenue is passed on to the link-supplier. Indeed, since $\lambda > 0$, we have

$$\sum_m \frac{1}{\beta_m} \left(\frac{2p_m}{\lambda + \sqrt{\lambda^2 + 4\frac{p_m}{\beta_m}}} \right)^2 < \sum_m p_m$$

where the right-hand side is obtained when $\lambda = 0$. The actions of the link-supplier as a strategic agent creates a situation of conflict and results in the following undesirable equilibrium.

Theorem 2: When the users and the link-supplier are price-anticipating, the only Nash equilibrium is $(\mathbf{p}^o, \boldsymbol{\beta}^o)$ where $p_m^o = 0$ and $\beta_m^o = 0$ for all $m = 1, 2, \cdots, M$.

Proof: See Appendix B-B. ∎

Thus, in the price-anticipating setting, efficiency loss is 100%, which we interpret as a market break-down. Indeed, at $\boldsymbol{\beta}^o = \mathbf{0}$, the link-supplier is assured an income of $\sum_m p_m$. Given this guaranteed income, he minimizes his cost by supplying zero capacity. The resulting equilibrium is one with the lowest efficiency, and the situation is vastly different from the setting when the link-supplier is not viewed as an agent [10].

V. Price-Anticipation with link as lead player

In view of the break-down of the market when both the users and the link-supplier are simultaneously price anticipating, we design an alternative scheme that involves an additional stage. The sequence of exchanges is as follows.

PRICE-ANTICIPATION WITH LINK AS LEADER (PALL)

1) The network manager first announces to the agents how the allocation will be done and the payments will be fixed, as a function of prices and agents' bids.
2) The link-supplier then announces his bid-vector $\boldsymbol{\beta}$. This information is made available to all users.
3) The users then send their bids $p_m^{\boldsymbol{\beta}}$ $(m = 1, 2, \cdots, M)$. Let $\boldsymbol{p}^{\boldsymbol{\beta}} = (p_1^{\boldsymbol{\beta}}, p_2^{\boldsymbol{\beta}}, \cdots, p_M^{\boldsymbol{\beta}})$.
4) The network-manager then computes the prices $(\boldsymbol{\mu}(\mathbf{p}^{\boldsymbol{\beta}},\boldsymbol{\beta}), \lambda(\mathbf{p}^{\boldsymbol{\beta}},\boldsymbol{\beta}))$ by solving the NETWORK problem in (2).
5) The payments and the rates-allocated are exactly as in the price-taking mechanism, but with $(\boldsymbol{\mu}, \lambda)$ replaced by $(\boldsymbol{\mu}(\mathbf{p}^{\boldsymbol{\beta}},\boldsymbol{\beta}), \lambda(\mathbf{p}^{\boldsymbol{\beta}},\boldsymbol{\beta}))$.

The analysis of this mechanism proceeds as follows. Given a $(\boldsymbol{\beta}, \boldsymbol{p})$, the expression for the prices set by the network-manager are as in Lemma 1. As a result, the expressions for the users' and the link-supplier's pay-off functions are exactly as in (12) and (13), respectively, but with $\mathbf{p}$ replaced by $\mathbf{p}^{\boldsymbol{\beta}}$. Using these pay-off functions, we characterize the solution in the form of Stackelberg equilibrium defined next.

Definition 3 (Stackelberg Equilibrium): A bid vector $(\boldsymbol{\beta}, \mathbf{p}^{\boldsymbol{\beta}})$ is a Stackelberg equilibrium if, for all $m = 1, 2, \cdots, M$, we have

$$\begin{aligned} Q_m(p_m^{\boldsymbol{\beta}}, \mathbf{p}_{-\mathbf{m}}^{\boldsymbol{\beta}}, \boldsymbol{\beta}) &\geq Q_m(\overline{p}_m, \mathbf{p}_{-\mathbf{m}}^{\boldsymbol{\beta}}, \boldsymbol{\beta}) \quad \forall \overline{p}_m \geq 0 \\ Q_L(\boldsymbol{\beta}, \mathbf{p}^{\boldsymbol{\beta}}) &\geq Q_L(\overline{\boldsymbol{\beta}}, \mathbf{p}^{\overline{\boldsymbol{\beta}}}) \quad \forall \overline{\boldsymbol{\beta}} \geq \mathbf{0}. \end{aligned}$$

□

Observe that the bid-vector $\boldsymbol{\beta}$ announced by the link-supplier in step-2 anticipates the user bids $\mathbf{p}^{\boldsymbol{\beta}}$ of step-3. For a given $\boldsymbol{\beta}$, the bids submitted by the users is in anticipation of the prices the network-manager announces in step-4.

For the ease of exposition, we assume that $C = \infty$ so that the capacity constraint is not binding (the case where C is finite can be similarly handled). Thus, recalling (9) and (11), we have $\lambda(\mathbf{p},\boldsymbol{\beta}) = 0$ and $\mu_m(\mathbf{p},\boldsymbol{\beta}) = \sqrt{\frac{p_m}{\beta_m}}$. As a result the pay-off functions can be simply expressed as

$$Q_m(p_m, \mathbf{p}_{-m}, \boldsymbol{\beta}) = U_m\left(\sqrt{p_m \beta_m}\right) - p_m \tag{14}$$

$$Q_L(\boldsymbol{\beta}, \mathbf{p}) = -V\left(\sum_m \sqrt{p_m \beta_m}\right) + \sum_m p_m. \tag{15}$$

This simplification will enable us to focus on the key ideas rather than dwell on the technicalities arising from a finite C (which can be handled but is cumbersome and not enlightening).

From (14) we see that the user pay-offs are independent of the bids submitted by the other users. As a result, for a given $\boldsymbol{\beta}$, the unique equilibrium strategy for user-m is given by

$$p_m^{\boldsymbol{\beta}} = \underset{p_m \geq 0}{\arg\max} \left(U_m\left(\sqrt{p_m \beta_m}\right) - p_m \right). \tag{16}$$

In Lemma 2 we report the expression for $p_m^{\boldsymbol{\beta}}$ that is obtained by solving (16).

Lemma 2: For a given $\boldsymbol{\beta}$ we have

$$p_m^{\boldsymbol{\beta}} = \begin{cases} \frac{r_{\beta_m}^2}{\beta_m} & \text{if } \beta_m > 0 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where r_{β_m} is the fixed point of $U_m'(r) = 2r/\beta_m$.

Proof: Since the objective function in (16) is continuously differentiable and strictly concave (both are easy to check), it suffices to show that $p_m^{\boldsymbol{\beta}}$ of (17) solves the following optimality equation:

$$U_m'\left(\sqrt{p_m \beta_m}\right) \frac{\sqrt{\beta_m}}{2\sqrt{p_m}} - 1 = 0.$$

Indeed, with $p_m = p_m^{\boldsymbol{\beta}}$ of (17) plugged into the above expression we have

$$U_m'(r_{\beta_m}) \frac{\beta_m}{2 r_{\beta_m}} - 1 = 0$$

and so r_{β_m} satisfies $U_m'(r_{\beta_m}) = 2r_{\beta_m}/\beta_m$. The case when $\beta_m = 0$ is straightforward. ■

We extend the definition of r_{β_m} in the above lemma by defining $r_{\beta_m} = 0$ if $\beta_m = 0$. It is then easy to see that $r_{\beta_m} = \sqrt{p_m^{\boldsymbol{\beta}} \beta_m}$ is the allocation to user m. Plugging the above result into (15), we compute the optimal $\boldsymbol{\beta}$ that the link-supplier should announce in step-1 as

$$\boldsymbol{\beta}^* \in \mathcal{B}^* = \arg\max_{\boldsymbol{\beta} \geq \mathbf{0}} \left\{ -V\left(\sum_m r_{\beta_m}\right) + \sum_m \frac{r_{\beta_m}^2}{\beta_m} \right\}, \tag{18}$$

where $\boldsymbol{\beta} \geq \mathbf{0}$ means component-wise inequality.

For any $\boldsymbol{\beta}^* \in \mathcal{B}^*$ it is clear that $(\boldsymbol{\beta}^*, \boldsymbol{p}^{\boldsymbol{\beta}^*})$ constitutes a Stackelberg equilibrium, where the rate allocated to user-m is given by $x_m^{\boldsymbol{\beta}^*} = y_m^{\boldsymbol{\beta}^*} = \sqrt{p_m^{\boldsymbol{\beta}^*} \beta_m^*} = r_{\beta_m^*}$. However, we first need to assert the existence of a solution $\boldsymbol{\beta}^*$, i.e., that the set $\mathcal{B}^*$ is nonempty.

Theorem 3: Suppose $U_m(\cdot)$ and $V(\cdot)$ satisfy the following: $xU_m'(x) \to \infty$ and $V(x)/x \to \infty$ as $x \to \infty$. Then the set $\mathcal{B}^*$ is nonempty. Hence, under the above assumptions on the pay-offs and cost function, a Stackelberg equilibrium exists.

Proof: See Appendix C. ■

Remark: The above assumption excludes cost functions that are asymptotically linear, and pay-offs such as $\log(1+x)$. However, we note that these assumptions are not too restrictive. Also, note that it is not possible to assert the uniqueness of β^* as it is not clear how $r_{\beta_m}^2/\beta_m$ varies as a function of β_m (although it can be shown that r_{β_m} increases with β_m).

In the remainder of this section, we restrict attention to linear user pay-offs.

A. Stackelberg Equilibrium for Linear User Pay-offs

An explicit expression for the Stackelberg equilibrium can be derived when the user pay-offs are linear. Suppose that the user pay-offs are of the form $U_m(x_m) = c_m x_m$ where $c_m > 0$ ($m = 1, 2, \cdots, M$). Without loss of generality, assume that $c_1 = \max_m\{c_m\}$. The Stackelberg equilibrium can then be computed as follows.

First, fix a $\boldsymbol{\beta}$. Recalling Lemma 2, we have

$$r_{\beta_m} = \frac{\beta_m U_m'(r_{\beta_m})}{2} = \frac{\beta_m c_m}{2}$$

so that the equilibrium bid of user-m can be written as

$$p_m^{\boldsymbol{\beta}} = \frac{r_{\beta_m}^2}{\beta_m} = \frac{\beta_m c_m^2}{4}. \tag{19}$$

Substituting for r_{β_m} in (18), the optimal $\boldsymbol{\beta}^*$ can be computed by solving

$$\max_{\boldsymbol{\beta} \geq \mathbf{0}} \left\{ -V\left(\sum_m \frac{\beta_m c_m}{2}\right) + \sum_m \frac{\beta_m c_m^2}{4} \right\}.$$

The solution to the above problem is given by

$$\beta_m^* = \begin{cases} \frac{2}{c_1} v^{-1}(\frac{c_1}{2}) & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

where $v(x) = V'(x)$. The equilibrium bids of users in response to this optimized $\boldsymbol{\beta}^*$ is then given by

$$p_m^{\boldsymbol{\beta}^*} = \begin{cases} \frac{c_1}{2} v^{-1}\left(\frac{c_1}{2}\right) & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Thus, when the user pay-offs are linear, the link-supplier allocates all the bandwidth to the "best" user (i.e., the one with the maximum slope c_m); in return, the best user alone makes a positive payment to the link-supplier.

The rate allocated to user m at equilibrium is

$$\begin{aligned} x_m^{\boldsymbol{\beta}^*} &= r_{\beta_m^*} \\ &= \sqrt{p_m^{\boldsymbol{\beta}^*} \beta_m^*} \\ &= \begin{cases} v^{-1}\left(\frac{c_1}{2}\right) & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{22}$$

The total rate served by the link-supplier at equilibrium is given by $\sum_m y_m^{\boldsymbol{\beta}^*} = \sum_m x_m^{\boldsymbol{\beta}^*} = v^{-1}\left(\frac{c_1}{2}\right)$.

B. Lower Bound on Efficiency for Linear User Pay-offs

Given a Stackelberg equilibrium $(\boldsymbol{\beta}^*, \boldsymbol{p}^{\boldsymbol{\beta}^*})$ the *efficiency* is defined as the ratio of the utility at equilibrium (*Stackelberg utility*) to the system optimum (*social utility*):

$$\mathcal{E}(\{U_m\}; V) = \frac{\sum_m U_m\left(x_m^{\boldsymbol{\beta}^*}\right) - V\left(\sum_m x_m^{\boldsymbol{\beta}^*}\right)}{\sum_m U_m\left(x_m^s\right) - V\left(\sum_m x_m^s\right)} \tag{23}$$

where x_m^s denotes the social optimum allocation to user m (obtained by solving SYSTEM in (1)). Note that we have emphasized the dependency of efficiency on $(\{U_m\}; V)$ by incorporating these into the notation for efficiency.

When the link-supplier is non-strategic, from Johari et al. [10] it is known that the bound on efficiency is $(4\sqrt{2}-5)$, i.e., $\mathcal{E}(\{U_m\}, V) \geq (4\sqrt{2}-5)$ for any general collection of user pay-off functions $\{U_m\}$ (the loss in efficiency is thus no more than 34%). The above bound is obtained in [10] by doing the following.

(a) Show that the users' equilibrium bids in the *original game* (with general user pay-off functions) constitutes an equilibrium in an *alternate game* with appropriately chosen linear pay-off functions.

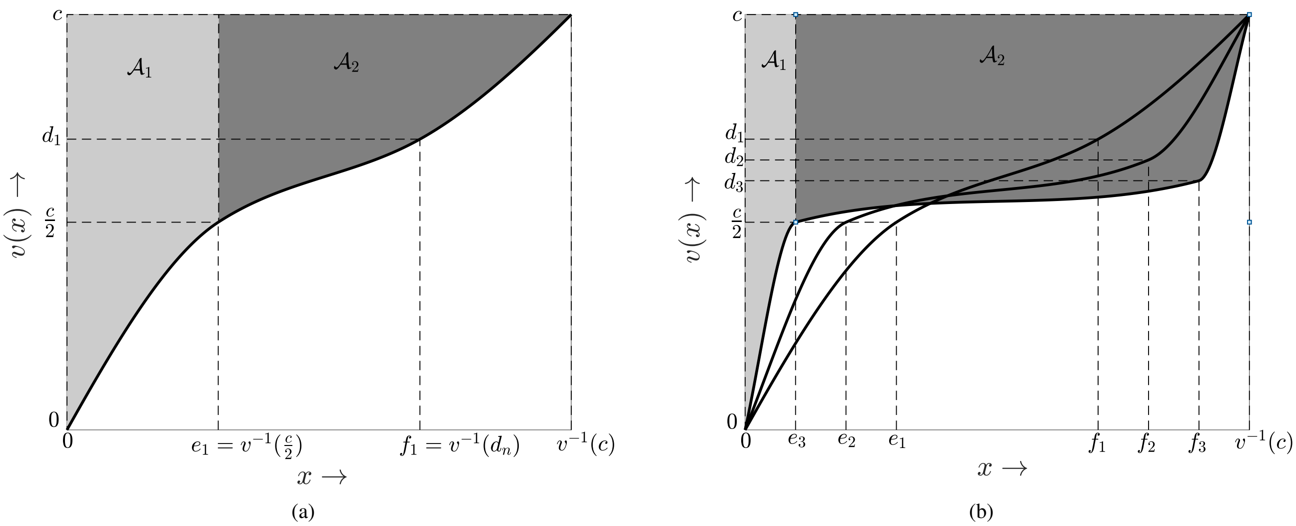

Fig. 1. Geometric interpretation for the efficiency bound.

(b) Use this to show that the efficiency in the original game is bounded below by the efficiency achieved in the alternate game.
(c) Finally, minimize the efficiency over the set of all linear pay-offs; this can be explicitly computed and is $(4\sqrt{2}-5)$.

In our case, although (a) holds[1] for any given $\boldsymbol{\beta}$, there is a subtle issue[2]. Since the link-supplier is also strategic, the original game and the alternate game (with linear user pay-offs) may not have identical Stackelberg equilibria. In particular, the $\boldsymbol{\beta}^*$ that optimizes the objective in (18) may not necessarily optimize

$$\max_{\boldsymbol{\beta} \geq \mathbf{0}} \left\{ -V\left(\sum_m \frac{\beta_m a_m}{2} \right) + \sum_m \frac{\beta_m a_m^2}{4} \right\}, \tag{24}$$

which is the objective corresponding to the game with linear pay-offs: $\overline{U}_m(x_m) = a_m x_m$ with $a_m = U'_m(r_{\beta^*_m})$. Thus, (a) and (b) may not hold for general user pay-offs. However, an analog of (c) continues to hold if we restrict our attention to the ensemble of all linear user pay-offs. The lower bound on efficiency will however depend on the link-suppliers cost function $V(x)$. This result is detailed in the following theorem.

Theorem 4: Fix a link-cost function $V(\cdot)$. For any set of linear user pay-offs $\{U_m\}$, we have

$$\mathcal{E}(\{U_m\}; V) \;\; \geq \;\; \inf_{c>0} \frac{cv^{-1}(\frac{c}{2}) - V(v^{-1}(\frac{c}{2}))}{cv^{-1}(c) - V(v^{-1}(c))} \tag{25}$$

where $v(\cdot) := V'(\cdot)$.

Proof: See Appendix D. ■

[1]Formally, we can show that for any given $\boldsymbol{\beta}$, the equilibrium strategy $\boldsymbol{p}_m^{\boldsymbol{\beta}}$ for the users in the original game with pay-off functions $\{U_m\}$ is also an equilibrium strategy for the users in an alternate game with linear pay-offs $\{\overline{U}_m\}$, where $\overline{U}_m(x_m) = c_m x_m$ with $c_m = U'_m(r_{\beta^*_m})$.

[2]Our conference version [1] missed this subtle point and incorrectly made a more general claim that the lower bound held for a larger class of user pay-offs.

C. *Efficiency Bound for Linear User Pay-offs and Polynomial Link-Costs*

We apply the above theorem to derive explicit expressions for the lower bound on the efficiency when the link-cost function is the polynomial bx^n. We start with the simplest case of quadratic link-cost, i.e., $V(x) = bx^2$ where $b > 0$. We then have $v(x) = 2bx$ so that $v^{-1}(y) = \frac{y}{2b}$. Thus, using (25), we obtain

$$\begin{aligned} \mathcal{E}(\{U_m\}; V) \;\; &\geq \;\; \inf_{c>0} \frac{c\frac{c}{4b} - V(\frac{c}{4b})}{c\frac{c}{2b} - V(\frac{c}{2b})} \\ &= \;\; \inf_{c>0} \frac{c\frac{c}{4b} - b(\frac{c}{4b})^2}{c\frac{c}{2b} - b(\frac{c}{2b})^2} \\ &= \;\; \inf_{c>0} \frac{\frac{c^2}{4b}(1-\frac{1}{4})}{\frac{c^2}{2b}(1-\frac{1}{2})} \\ &= \;\; \frac{3}{4}. \end{aligned}$$

Thus, when the link-cost is quadratic, the worst-case efficiency loss for any linear user pay-off is no more that 25%.

Similarly, suppose $V(x) = bx^3$ for $x \geq 0$, with $b > 0$. (This is increasing and convex for $x \geq 0$.) Then, using the bound (25) and a similar calculation, we obtain

$$\mathcal{E}(\{U_m\}; V) \;\; \geq \;\; \frac{5}{4\sqrt{2}} \;\; \geq \;\; 0.88.$$

Thus, the worst-case efficiency loss improves to 12% when the link-cost is cubic. In general, suppose the link-cost is polynomial of degree $n \geq 2$, i.e., $V(x) = bx^n$, $x \geq 0, b > 0$, then the bound on efficiency is given by

$$\mathcal{E}(\{U_m\}; V) \;\; \geq \;\; \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \frac{2n-1}{n-1}. \tag{26}$$

The aforementioned lower bound is increasing as a function of n and converges to 1 as $n \rightarrow \infty$. Thus, if the link-cost can be modeled as bx^n, the efficiency loss reduces with increase in n.

The above observation provides strong support for our proposed PALL mechanism when compared with the price-anticipating mechanism of Section IV whose efficiency loss (for any $\{U_m\}$ including linear user pay-offs and any V) is 100%.

D. Worst-Case Bound on Efficiency for Linear User Pay-offs

Although the class of polynomial link-cost functions yield favorable lower bounds on efficiency, we now show that there exists a family of link-cost functions V_n, $n \geq 1$, such that the corresponding sequence of efficiency-bound converges to 0 as $n \to \infty$. Thus, the *worst-case* efficiency bound, over all possible linear $\{U_m\}$ and over *all possible* V, is 0.

To see this, let us first rewrite (25) by expressing V in the integral form $V(x) = \int_0^x v(\tau)d\tau$ to get

$$\begin{aligned}\mathcal{E}(\{U_m\}; V) \quad &\geq \quad \inf_{c>0} \frac{cv^{-1}(\frac{c}{2}) - \int_0^{v^{-1}(\frac{c}{2})} v(\tau)d\tau}{cv^{-1}(c) - \int_0^{v^{-1}(c)} v(\tau)d\tau} \\ &=: \quad \inf_{c>0} H(c, v).\end{aligned}$$

For a given c and a marginal cost function for the link-supplier $v(\cdot)$, $H(c, v)$ can be geometrically interpreted with the aid of the illustration in Fig.1(a) as follows: the numerator in the formula for efficiency is the area of the region $\mathcal{A}_1$ (light shaded region) while the denominator is total area of $\mathcal{A}_1$ and $\mathcal{A}_2$ (shaded dark). We then have

$$H(c, v) = \frac{A_1}{A_1 + A_2} = \frac{A_1/A_2}{1 + A_1/A_2}$$

where A_i denotes the area of region $\mathcal{A}_i$ ($i = 1, 2$). In Fig. 1(a) we have used e_1 to denote $v^{-1}(\frac{c}{2})$; also, $f_1 = v^{-1}(d_1)$ where d_1 is arbitrarily chosen in $\left(\frac{c}{2}, c\right)$. Since V is strictly convex and increasing, it follows that v is strictly increasing.

Now, it is possible to construct a sequence of $v(\cdot)$ functions, say $\{v_n\}$, such that $e_n \downarrow 0$, $d_n \downarrow \frac{c}{2}$ while $f_n \uparrow v^{-1}(c)$; an illustration of such a construction is depicted in Fig. 1(b). Observe that along such a sequence we have $A_1 \downarrow 0$ and $A_2 \uparrow \frac{c}{2}v^{-1}(c) > 0$. As a result we have $H(c, v_n) \to 0$ as $n \to \infty$. Thus, for any given c it is possible to produce pathological link-cost functions whose efficiency-bounds are arbitrarily close to 0. Therefore, it is not possible to guarantee a less-than-100% efficiency loss (i.e., a positive efficiency) when the class of all possible link-cost functions are considered. Nevertheless, bounding the efficiency for a fixed link-cost function is reassuring.

PART II: MULTIPLE LINKS

VI. SYSTEM MODEL AND PROBLEM DEFINITION

In this section we extend our results to the more general setting with multiple links. We assume an example scenario with parallel links so that the users have the flexibility to off-load different amounts of rates on different links. Simultaneously, the respective link-managers have to be competitive in terms of their bids in order to maximize their respective pay-offs[3]. Although it is natural to expect active participation from both users and link-managers, in the upcoming Theorem 6 we show the contrary. We will see that, when the users and the link-managers are strategic, the market collapses due to zero participation from both types of agents. This outcome is similar to the single-link case. This also establishes that the break-down in the single-link case is not due to the monopolistic nature of the supplier in the single-link setting. In Theorem 6, alternative routes exist, and yet, the undesirable equilibrium ensues.

[3]Extension to a general network as in Kelly [13] is straightforward and does not bring out any new phenomenon.

We begin by generalizing our notation from Section II. As before we assume that there are M users in the system. However, we now generalize our earlier model by introducing L parallel links. The capacity of link $\ell = 1, 2, \cdots, L$ is given by $C_\ell > 0$. Let $x_{m\ell}$ denote the rate requested by user m on link ℓ, and let $y_{m\ell}$ be the rate the link-manager ℓ is willing to allocate to user m. Thus, $\mathbf{x}_m = (x_{m1}, x_{m2}, \cdots, x_{mL})$ is the *rate-request vector* of user m, and $\mathbf{y}_\ell = (y_{1\ell}, y_{2\ell}, \cdots, y_{M\ell})$ is the *rate-allocation vector* of link ℓ. Let $\mathbf{X} = (\mathbf{x}_m : m = 1, 2, \cdots, M)$ and $\mathbf{Y} = (\mathbf{y}_\ell : \ell = 1, 2, \cdots, L)$ denote the rate-request matrix and rate-allocation matrix, respectively. The user pay-off and the link-cost functions are given by U_m and V_ℓ. As before, we assume that U_m and V_ℓ are concave and strictly convex, respectively. In addition, both U_m and V_ℓ are strictly increasing and continuously differentiable with $U'_m(0)$ finite.

The analog of the problem SYSTEM in (1) is given by (in the sequel, the acronym ML stands for Multi-Link):

ML-SYSTEM

$$\text{Maximize: } \sum_m U_m\left(\sum_\ell x_{m\ell}\right) - \sum_\ell V_\ell\left(\sum_m y_{m\ell}\right) \quad (27a)$$

$$\text{Subject to: } \sum_m y_{m\ell} \leq C_\ell \ \forall \ell \quad (27b)$$

$$x_{m\ell} \leq y_{m\ell}, x_{m\ell} \geq 0, y_{m\ell} \geq 0 \ \forall m, \ell. \quad (27c)$$

Similarly, denoting the users' and the link-managers' bid-vectors as

$$\begin{aligned}\mathbf{p}_m &= (p_{m1}, p_{m2}, \cdots, p_{mL}) \\ \boldsymbol{\beta}_\ell &= (\beta_{1\ell}, \beta_{2\ell}, \cdots, \beta_{M\ell}),\end{aligned}$$

respectively, the analog of problem NETWORK in (2) is:

ML-NETWORK

$$\text{Maximize: } \sum_{m,\ell}\left(p_{m\ell}\log(x_{m\ell}) - \frac{y_{m\ell}^2}{2\beta_{m\ell}}\right) \quad (28a)$$

$$\text{Subject to: } \sum_m y_{m\ell} \leq C_\ell \ \forall \ell \quad (28b)$$

$$x_{m\ell} \leq y_{m\ell}, x_{m\ell} \geq 0, y_{m\ell} \geq 0 \ \forall m, \ell. \quad (28c)$$

We introduce some more notation. Let $\mathbf{P} = (\mathbf{p}_m : m = 1, 2, \cdots, M)$ denote the users' bid matrix. Similarly, the link-managers' bid matrix is denoted by $\mathbf{B} = (\boldsymbol{\beta}_\ell : \ell = 1, 2, \cdots, L)$. The network-manager sets prices $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_L)$ and $\mathbf{M} = (\boldsymbol{\mu}_m : m = 1, 2, \cdots, M)$ where $\boldsymbol{\mu}_m = (\mu_{m1}, \mu_{m2}, \cdots, \mu_{mL})$. The prices $\boldsymbol{\lambda}$ and $\mathbf{M}$ are essentially the Lagrange multipliers associated with the constraints (28b) and (28c), respectively.

We investigate the price-taking and the price-anticipating scenarios separately, as was done in the single-link setting.

VII. Price-Taking Scenario

The mechanism under the price-taking scenario is exactly as in Section III (see PTM in Section III), except that now there are multiple link-managers who submit their respective bids $\boldsymbol{\beta}_\ell$ ($\ell = 1, 2, \cdots, L$) simultaneously. In this setting, given the prices $(\boldsymbol{\lambda}, \mathbf{M})$ set by the network-manager, the pay-off to user m can be written as

$$P_m(\boldsymbol{p}_m; \boldsymbol{\mu}_m) = U_m\left(\sum_\ell \frac{p_{m\ell}}{\mu_{m\ell}}\right) - \sum_\ell p_{m\ell}. \quad (29)$$

Similarly, the pay-off to the link-manager ℓ is given by

$$P_{L,\ell}\Big(\boldsymbol{\beta}_\ell; (\boldsymbol{\mu}_\ell, \lambda_\ell)\Big) = -V_\ell\left(\sum_m \beta_{m\ell}\Big(\mu_{m\ell} - \lambda_\ell\Big)\right) + \sum_m \beta_{m\ell}\Big(\mu_{m\ell} - \lambda_\ell\Big)^2 \quad (30)$$

where $\boldsymbol{\mu}_\ell := (\mu_{1,\ell}, \mu_{2,\ell}, \cdots, \mu_{M,\ell})$. The following are the generalizations of Definition 1 and Theorem 1, respectively.

Definition 4 (Competitive Equilibrium): A vector of bids and prices $(\boldsymbol{P}, \mathbf{B}, \boldsymbol{\lambda}, \mathbf{M})$ is said to constitute a competitive equilibrium if the following conditions hold:

(C1) $P_m(\mathbf{p}_m; \boldsymbol{\mu}_m) \geq P_m(\overline{\mathbf{p}}_m; \boldsymbol{\mu}_m)\ \forall \overline{\mathbf{p}}_m \geq \mathbf{0}, \forall m$

(C2) $P_{L,\ell}\Big(\boldsymbol{\beta}_\ell; (\boldsymbol{\mu}_\ell, \lambda_\ell)\Big) \geq P_{L,\ell}\Big(\overline{\boldsymbol{\beta}}_\ell; (\boldsymbol{\mu}_\ell, \lambda_\ell)\Big) \forall \overline{\boldsymbol{\beta}}_\ell \geq \mathbf{0}, \forall \ell$

(C3) For each ℓ define $\mathcal{M}_\ell = \Big\{m : \mu_{m\ell} \neq \lambda_\ell\Big\}$ and $\widehat{C}_\ell = \sqrt{\left(\sum_m p_{m\ell}\right)\left(\sum_{m \in \mathcal{M}_\ell} \beta_{m\ell}\right)}$. Then,

- **(C3-a)** $\frac{p_{m\ell}}{\mu_{m\ell}} = \beta_{m\ell}\Big(\mu_{m\ell} - \lambda_\ell\Big)\ \forall (m, \ell)$;
- **(C3-b)** $\mu_{m\ell} = \mu(\ell)$ where
$$\mu(\ell) = \sum_i p_{i\ell} \Big/ \min\Big\{C_\ell, \widehat{C}_\ell\Big\}\ \forall \ell, \forall m \in \mathcal{M};$$
- **(C3-c)** $\lambda_\ell = \min\left\{0, \left(1 - \left(\frac{C_\ell}{\widehat{C}_\ell}\right)^2\right)\frac{\sum_i p_{i\ell}}{C_\ell}\right\}\ \forall \ell.$

□

Theorem 5: When the users and the link-managers are price-taking, there exists a competitive equilibrium. Moreover, given a competitive equilibrium $(\mathbf{P}, \boldsymbol{B}, \boldsymbol{\lambda}, \mathbf{M})$, the rate matrices $\mathbf{X}$ and $\mathbf{Y}$, defined as $x_{m\ell} = p_{m\ell}/\mu_{m\ell}$ and $y_{m\ell} = \beta_{m\ell}\Big(\mu_{m\ell} - \lambda_\ell\Big)\ \forall (m, \ell)$, are optimal for the problem ML-SYSTEM in (27).

Proof: The proof is omitted since it is a straightforward extension of the proof of Theorem 1. ■

VIII. Price-Anticipating Scenario

Recall that when the users and the link-managers are price-anticipating they expect that the bids submitted by them affect the prices set by the network-manager. In particular, the users and the link-managers are aware that the prices $(\boldsymbol{\lambda}(\mathbf{P}, \mathbf{B}), \mathbf{M}(\mathbf{P}, \mathbf{B}))$ set by the network-manager, in response to the bids $(\mathbf{P}, \mathbf{B})$ submitted by the agents, are dual-optimal for the problem ML-NETWORK in (28). The details of the mechanism under price-anticipating scenario is similar to PAM in Section IV, except that the setting now consists of multiple link-managers who submit their respective bids simultaneously.

Now, the expressions for the prices set by the network-manager is as reported in the following lemma (which is in-line with the result in Lemma 1).

Lemma 3: Given any matrix $(\mathbf{P}, \mathbf{B})$ of users' and link-managers' bids, the prices $(\boldsymbol{\lambda}(\mathbf{P}, \mathbf{B}), \mathbf{M}(\mathbf{P}, \mathbf{B}))$ set by the network-manager are given by, $\forall (\ell, m)$

$$\lambda_\ell(\mathbf{P}, \mathbf{B}) = \begin{cases} 0 & \text{if } \sum_i \sqrt{p_{i\ell}\beta_{i\ell}} \leq C_\ell \\ f_\ell^{-1}(C_\ell) & \text{otherwise,} \end{cases}$$

where f_ℓ^{-1} denotes the inverse of the function f_ℓ which is defined as

$$f_\ell(t) = \sum_i \left(\frac{2p_{i\ell}}{t + \sqrt{t^2 + 4\frac{p_{i\ell}}{\beta_{i\ell}}}}\right),$$

and

$$\mu_{m\ell}(\mathbf{P}, \mathbf{A}) = \frac{\lambda_\ell(\mathbf{P}, \mathbf{A}) + \sqrt{\lambda_\ell(\mathbf{P}, \mathbf{A})^2 + 4\frac{p_{m\ell}}{\beta_{m\ell}}}}{2}.$$

Proof: Omitted since it is a straightforward extension of the proof of Lemma 1. ■

The pay-off to the users in the price-anticipating scenario can thus be expressed as

$$Q_m(\mathbf{p}_m, \mathbf{p}_{-m}, \mathbf{B}) = U_m\left(\sum_\ell \frac{p_{m\ell}}{\mu_{m\ell}(\mathbf{P}, \mathbf{B})}\right) - \sum_\ell p_{m\ell}. \quad (31)$$

It is possible to simplify the above expression by substituting for $\mu_{m\ell}(\mathbf{P}, \mathbf{B})$. We then obtain an expression that is a generalization of the pay-off function in (12) (note that $Q_m(\cdot)$ in (12) is for $L = 1$). However, we now need to consider 2^L sub-cases depending on whether $\sum_i \sqrt{p_{i\ell}\beta_{i\ell}} \leq C_\ell$ or otherwise, for each $\ell = 1, 2, \cdots, L$. For instance, for $L = 2$ there are 4 possible sub-cases. Suppose $\sum_i \sqrt{p_{i1}\beta_{i1}} \leq C_1$ and $\sum_i \sqrt{p_{i2}}\beta_{i2} > C_2$. Then, denoting $\lambda_2 := \lambda_2(\mathbf{P}, \mathbf{A})$, we have

$$Q_m(\mathbf{p}_m, \mathbf{p}_{-m}, \mathbf{B}) = U_m\left(\sqrt{p_{m1}\beta_{m1}} + \frac{2p_{m2}}{\lambda_2 + \sqrt{\lambda_2^2 + 4\frac{p_{m2}}{\beta_{m2}}}}\right) - \sum_{\ell=1}^{2} p_{m\ell}.$$

The expression for the links' pay-off function, however, comprises only two sub-cases as in (13):

$$Q_{L,\ell}(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell}, \mathbf{P}) = \begin{cases} -V_\ell\left(\sum_m \sqrt{p_{m\ell}\beta_{m\ell}}\right) + \sum_m p_{m\ell} & \text{if } \sum_i \sqrt{p_{i\ell}\beta_{i\ell}} \leq C_\ell \\ -V_\ell(C_\ell) + \sum_m \frac{1}{\beta_{m\ell}}\left(\frac{2p_{m\ell}}{\lambda_\ell + \sqrt{\lambda_\ell^2 + 4\frac{p_{m\ell}}{\beta_{m\ell}}}}\right)^2 & \text{otherwise} \end{cases} \quad (32)$$

where $\lambda_\ell := \lambda_\ell(\mathbf{A}, \mathbf{P})$. Using the above pay-off functions, the definition of Nash equilibrium in Definition 2 can be analogously extended to bid vectors $(\mathbf{P}, \mathbf{B})$ in the multiple-link setting.

Finally, in the following theorem we report the existence and uniqueness of the Nash equilibrium $(\mathbf{P}^o, \mathbf{B}^o)$, which is inefficient in the sense that the market is non-functional at $(\mathbf{P}^o, \mathbf{B}^o)$ with users making zero payments (i.e., $p_{m\ell} = 0$) and the link-managers providing zero rate (i.e., $\beta_{m\ell} = 0$).

Theorem 6: When the users and the link-managers are price-anticipating, the only Nash equilibrium is $(\mathbf{P}^o, \mathbf{B}^o)$ where $p^o_{m\ell} = 0$ and $\beta^o_{m\ell} = 0 \ \forall m, \ell$.

Proof: See Appendix E. ■

IX. Price-Anticipation with Links as Lead Players

We begin as in Section V. The network-manager announces the allocation procedure and the payment determination function. The link-suppliers then choose their bid-vectors $\boldsymbol{\beta}_\ell$ ($\ell = 1, 2, \cdots, L$). Following this the users choose their bid-vectors $\mathbf{p}^{\mathbf{B}}_m$ ($m = 1, 2, \cdots, M$) where $\mathbf{B} = (\boldsymbol{\beta}_\ell : \ell = 1, 2, \cdots, L)$ denotes the link-bid matrix. Let $\mathbf{P}^{\mathbf{B}}$ denote the user bid-matrix. The mechanism under this scenario is exactly as detailed under PALL in Section V, except that now there are multiple link-managers who announce their respective bids $\boldsymbol{\beta}_\ell$ simultaneously in Step-2. We then have the following generalization of the Stackelberg equilibrium.

Definition 5 (Stackelberg Equilibrium): A bid matrix $(\mathbf{B}, \mathbf{P}^{\mathbf{B}})$ is said to constitute a Stackelberg equilibrium for the game $\mathcal{G}(\{U_m\}, \{V_\ell\})$ if, for all $m = 1, 2, \cdots, M$ and $\ell = 1, 2, \cdots, L$, we have

$$\begin{aligned} Q_m(\mathbf{p}^{\boldsymbol{B}}_m, \mathbf{p}^{\mathbf{B}}_{-\mathbf{m}}, \mathbf{B}) &\geq Q_m(\overline{\mathbf{p}}_m, \mathbf{p}^{\boldsymbol{B}}_{-m}, \mathbf{B}) \quad \forall \overline{\mathbf{p}}_m \geq 0 \\ Q_{L,\ell}(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell}, \mathbf{P}^{\mathbf{B}}) &\geq Q_{L,\ell}(\overline{\boldsymbol{\beta}}_\ell, \boldsymbol{\beta}_{-\ell}, \mathbf{P}^{\overline{\mathbf{B}}}) \quad \forall \overline{\boldsymbol{\beta}} \geq \mathbf{0} \end{aligned}$$

where $\overline{\mathbf{B}} = [\overline{\boldsymbol{\beta}}_\ell \ \boldsymbol{\beta}_{-\ell}]$ is the bid matrix that results when link ℓ unilaterally deviates from $\boldsymbol{\beta}_\ell$ to $\overline{\boldsymbol{\beta}}_\ell$. □

Again, as in Section V, for simplicity we relax the capacity constraint by assuming that $C_\ell = \infty$ for all $\ell = 1, 2, \cdots, L$. We then have, for all m and ℓ

$$Q_m(\mathbf{p}_m, \mathbf{p}_{-m}, \mathbf{B}) = U_m\left(\sum_\ell \sqrt{p_{m\ell}\beta_{m\ell}}\right) - \sum_\ell p_{m\ell} \quad (33)$$

$$Q_{L,\ell}(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell}, \mathbf{P}) = -V_\ell\left(\sum_m \sqrt{p_{m\ell}\beta_{m\ell}}\right) + \sum_m p_{m\ell}. \quad (34)$$

From (33) we see that the pay-off of user-m is completely decoupled from $\mathbf{p}_{-m}$, the pay-off of other users. As a result, given $\mathbf{B}$, the equilibrium strategy of user-m can be simply expressed as

$$\mathbf{p}^{\mathbf{B}}_m = \arg\max_{\mathbf{p}_m}\left(U_m\left(\sum_\ell \sqrt{p_{m\ell}\beta_{m\ell}}\right) - \sum_\ell p_{m\ell}\right). \quad (35)$$

The following lemma is then a generalization Lemma 2.

Lemma 4: For a given $\mathbf{B}$ we have, for all (m, ℓ)

$$p^{\mathbf{B}}_{m\ell} = \begin{cases} \frac{\beta_{m\ell} r_m(\mathbf{B})^2}{\left(\sum_k \beta_{mk}\right)^2} & \text{if } \sum_k \beta_{mk} > 0 \\ 0 & \text{otherwise,} \end{cases} \quad (36)$$

where $r_m(\mathbf{B}) = r_m(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell})$ is the solution to the equation $U'_m(r) = 2r/\left(\sum_k \beta_{mk}\right)$.

Proof: The optimality equation for the problem in (35) is given by

$$\begin{aligned} U'_m\left(\sum_k \sqrt{p_{mk}\beta_{mk}}\right)\frac{\sqrt{\beta_{m\ell}}}{2\sqrt{p_{m\ell}}} - 1 = 0 \qquad & \text{if } \beta_{m\ell} > 0 \\ p_{m\ell} = 0 \qquad & \text{if } \beta_{m\ell} = 0. \end{aligned}$$

It is easy to check that $p^{\mathbf{B}}_{m\ell}$ in (36) satisfies the above conditions, thus verifying first part of the lemma. For the second part, note that $\sum_k \beta_{mk} = 0$ implies $\beta_{m\ell} = 0$ for all ℓ (since $\beta_{m\ell}$ are non-negative). Thus, the problem in (35) reduces to $\mathbf{p}^{\mathbf{B}}_m = \arg\max_{\mathbf{p}_m}\left(-\sum_\ell p_{m\ell}\right)$, the (non-negative) solution to which is simply given by $p^{\mathbf{B}}_{m\ell} = 0$ for all ℓ. ■

Substituting for $\mathbf{P}^{\mathbf{B}}$ in (34), the link pay-off functions can be expressed as

$$\begin{aligned} S_\ell(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell}) &:= Q_{L,\ell}(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell}, \mathbf{P}^{\mathbf{B}}) \\ &= -V_\ell\left(\sum_m \sqrt{p^{\mathbf{B}}_{m\ell}\beta_{m\ell}}\right) + \sum_m p^{\mathbf{B}}_{m\ell} \\ &= -V_\ell\left(\sum_m \frac{\beta_{m\ell} r_m(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell})}{\sum_k \beta_{mk}}\right) + \sum_m \frac{\beta_{m\ell} r_m(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell})^2}{\left(\sum_k \beta_{mk}\right)^2} \end{aligned} \quad (37)$$

for $\ell = 1, 2, \cdots, L$. We denote the game played by the links alone with the above pay-off functions as $\mathcal{G}_L(\{U_m\}, \{V_\ell\})$, while the Stackelberg game played by all agents is denoted $\mathcal{G}(\{U_m\}, \{V_\ell\})$.

Definition 6: $\mathbf{B}^* = [\boldsymbol{\beta}^*_\ell \, \boldsymbol{\beta}^*_{-\ell}]$ is said to constitute a Nash equilibrium for the game $\mathcal{G}_L(\{U_m\}, \{V_\ell\})$ if

$$S_\ell(\boldsymbol{\beta}^*_\ell, \boldsymbol{\beta}^*_{-\ell}) \geq S_\ell(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}^*_{-\ell}).$$

for all $\boldsymbol{\beta}_\ell \geq \mathbf{0}$ and $\ell = 1, 2, \cdots, L$. □

The following lemma is straightforward.

Lemma 5: If $\mathbf{B}^*$ is a Nash equilibrium for the game $\mathcal{G}_L(\{U_m\}, \{V_\ell\})$ then $(\mathbf{B}^*, \mathbf{P}^{\mathbf{B}^*})$ is a Stackelberg equilibrium for the game $\mathcal{G}(\{U_m\}, \{V_\ell\})$.

Unlike the single-link case (recall Lemma 3), existence of a Stackelberg equilibrium for the game $\mathcal{G}(\{U_m\}, \{V_\ell\})$ under general user pay-offs is still open. This is because existence of a (pure strategy) Nash equilibrium for the game $\mathcal{G}_L(\{U_m\}, \{V_\ell\})$ is still open[4].

In the special case of linear user pay-offs, $U_m(x_m) = c_m x_m$, the pay-off functions in (37) for the game $\mathcal{G}_L(\{U_m\}, \{V_\ell\})$ get decoupled from the actions of other players. The analysis is greatly simplified and one can assert that a (pure strategy) Nash equilibrium exists for the game $\mathcal{G}_L(\{U_m\}, \{V_\ell\})$, and by Lemma 5, a corresponding Stackelberg equilibrium in the sense of Definition 5 exists. The next few subsections are dedicated to showing existence of the Stackelberg equilibrium for linear user pay-offs, lower bounding of the resulting the efficiency, and evaluating the worst-case efficiency. These generalise the results in Part I to the case of multiple links with linear user pay-offs.

[4]Tian [14, Th. 3.1] provides a necessary and sufficient condition for existence of pure strategy Nash equilibria in games with compact action spaces. We have not been able to verify whether Tian's condition holds for the game $\mathcal{G}_L(\{U_m\}, \{V_\ell\})$.

A. Existence of Stackelberg Equilibrium in the Case of Linear User Pay-offs

Consider linear pay-off functions $\{U_m\}$, where $U_m(x_m) = c_m x_m$ with $a_m > 0$. Now, given a bid matrix $\mathbf{B} = [\boldsymbol{\beta}_\ell,\ \boldsymbol{\beta}_{-\ell}]$, using Lemma 36 we can write

$$r_m(\mathbf{B}) = \frac{\left(\sum_k \beta_{mk}\right) U'_m(r_m(\mathbf{B}))}{2} = \frac{\left(\sum_k \beta_{mk}\right) c_m}{2}. \quad (38)$$

Substituting the above in (37) simplifies the link pay-off functions:

$$S_\ell(\boldsymbol{\beta}_\ell, \boldsymbol{\beta}_{-\ell}) = -V_\ell\left(\sum_m \frac{\beta_{m\ell} c_m}{2}\right) + \sum_m \frac{\beta_{m\ell} c_m^2}{4}$$

Thus, when the pay-offs are linear, the pay-off of link-manager ℓ depends solely on the action $\boldsymbol{\beta}_\ell = (\beta_{m\ell} : m = 1, \cdots, M)$ chosen by him. Thus, for $\mathbf{B}^* = [\boldsymbol{\beta}^*_\ell,\ \boldsymbol{\beta}^*_{-\ell}]$ to be a Nash equilibrium the following should hold:

$$\boldsymbol{\beta}^*_\ell \in \arg\max_{\boldsymbol{\beta}_\ell \geq \mathbf{0}} \left\{-V_\ell\left(\sum_m \frac{\beta_{m\ell} c_m}{2}\right) + \sum_m \frac{\beta_{m\ell} c_m^2}{4}\right\}. \quad (39)$$

for all $\ell = 1, 2, \cdots, L$.

Without loss of generality, assume that $c_1 = \max_m\{c_m\}$. Then, the solution to the above problem is given by

$$\beta^*_{m\ell} = \begin{cases} \frac{2}{c_m} v_\ell^{-1}\left(\frac{c_m}{2}\right) & \text{if } m = 1 \\ 0 & \text{otherwise,} \end{cases} \quad (40)$$

for all (m, ℓ), where $v_\ell(x) := V'_\ell(x)$. Substituting for $\beta^*_{m\ell}$ in (36) yields

$$p^{\mathbf{B}^*}_{m\ell} = \begin{cases} \frac{c_m}{2} v_\ell^{-1}\left(\frac{c_m}{2}\right) & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (41)$$

Thus, the rate allocated to user m on link-ℓ at equilibrium is given by

$$\begin{aligned} x^{\mathbf{B}^*}_{m\ell} &= \sqrt{p^{\mathbf{B}^*}_{m\ell} \beta^{\mathbf{B}^*}_{m\ell}} \\ &= \begin{cases} v_\ell^{-1}\left(\frac{c_m}{2}\right) & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (42)$$

Further, recalling (38), the total rate allocated to user m at equilibrium can be written as

$$\begin{aligned} \sum_\ell x^{\mathbf{B}^*}_{m\ell} &= \sum_\ell \sqrt{p^{\mathbf{B}^*}_{m\ell} \beta^{\mathbf{B}^*}_{m\ell}} \\ &= r_m(\mathbf{B}^*) \\ &= \begin{cases} \sum_\ell v_\ell^{-1}\left(\frac{c_m}{2}\right) & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (43)$$

Similarly, the total rate served by the link-manager ℓ at equilibrium is given by

$$\sum_m y^{\mathbf{B}^*}_{m\ell} = \sum_m \sqrt{p^{\mathbf{B}^*}_{m\ell} \beta^{\mathbf{B}^*}_{m\ell}} = v_\ell^{-1}\left(\frac{c_1}{2}\right) \quad (44)$$

The foregoing establishes that a Stackelberg equilibrium exists for the game $\mathcal{G}(\{U_m\}, \{V_\ell\})$ where the U_m are linear pay-offs.

B. Bound on Efficiency for Linear User Pay-offs

As in Part I we define the efficiency of a Stackelberg equilibrium $(\mathbf{B}^*, \mathbf{P}^{\mathbf{B}^*})$ as

$$\mathcal{E}(\{U_m\}; \{V_\ell\}) = \frac{\sum_m U_m\left(\sum_\ell x^{\mathbf{B}^*}_{m\ell}\right) - \sum_\ell V_\ell\left(\sum_m x^{\mathbf{B}^*}_{m\ell}\right)}{\sum_m U_m\left(\sum_\ell x^s_{m\ell}\right) - V\left(\sum_m x^s_{m\ell}\right)} \quad (45)$$

where $\{x^{\mathbf{B}^*}_{m\ell}\}$ denotes the rates allocated at Stackelberg equilibrium while $x^s_{m\ell}$ are the social optimum rates (obtained by solving ML-SYSTEM in (27)).

The following result is a generalization of Theorem 4.

Theorem 7: Fix a collection of link-cost functions $\{V_\ell(\cdot)\}$. For any set of linear user pay-offs $\{U_m\}$, we have

$$\mathcal{E}(\{U_m\}; \{V_\ell\}) \geq \inf_{c>0} \frac{\sum_\ell \left(c v_\ell^{-1}(\frac{c}{2}) - V_\ell(v_\ell^{-1}(\frac{c}{2}))\right)}{\sum_\ell \left(c v_\ell^{-1}(c) - V_\ell(v_\ell^{-1}(c))\right)} \quad (46)$$

where $v_\ell(\cdot) := V'_\ell(\cdot)$ for $\ell = 1, 2, \cdots, L$.

Proof: See Appendix F. ∎

In the next two subsections, we examine how low this efficiency can sink for polynomial link costs and for general link costs, under linear user pay-offs. As before, we have an optimistic result for polynomial link costs.

C. Efficiency Bound for Polynomial Link Costs

Applying Theorem 7 to polynomial link-cost functions, we obtain results analogous to those in Section V-C. For instance, suppose all the link-cost functions are quadratic, i.e., $V_\ell(x) = b_\ell x^2$ where $b_\ell > 0$ for $\ell = 1, 2, \cdots, L$. In this case we have, $v_\ell(x) = 2b_\ell x$ so that $v_\ell^{-1}(y) = \frac{y}{2b_\ell}$. Thus, the bound on efficiency can be written as

$$\begin{aligned} \mathcal{E}(\{U_m\}; \{V_\ell\}) &\geq \inf_{c>0} \frac{\sum_\ell \left(c\frac{c}{4b_\ell} - V_\ell(\frac{c}{4b_\ell})\right)}{\sum_\ell \left(c\frac{c}{2b_\ell} - V_\ell(\frac{c}{2b_\ell})\right)} \\ &= \inf_{c>0} \frac{\sum_\ell \left(c\frac{c}{4b_\ell} - b_\ell(\frac{c}{4b_\ell})^2\right)}{\sum_\ell \left(c\frac{c}{2b_\ell} - b_\ell(\frac{c}{2b_\ell})^2\right)} \\ &= \inf_{c>0} \frac{\sum_\ell \frac{c^2}{4b_\ell}(1 - \frac{1}{4})}{\sum_\ell \frac{c^2}{2b_\ell}(1 - \frac{1}{2})} \\ &= \frac{3}{4}. \end{aligned}$$

The above bound is identical to that obtained for the single link case in Part I.

Similarly, the bound on efficiency when the link costs are cubic (i.e., $V_\ell(x) = b_\ell x^3$) is $\mathcal{E}(\{U_m\}; \{V_\ell\}) \geq \frac{5}{4\sqrt{2}}$. In general, for polynomial link costs of the form $V_\ell(x) = b_\ell x^n$ ($n \geq 2$) we have

$$\mathcal{E}(\{U_m\}; \{V_\ell\}) \geq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \frac{2n-1}{n-1}.$$

which converges to 1 as $n \to \infty$. Thus, even in the multi-link setting, the efficiency approaches 1 when link cost functions bx^n are considered and $n \to \infty$.

D. Worst-Case Bound on Efficiency for Linear User Pay-offs

Again, as in Part I, the worst-case bound on efficiency can be arbitrarily close to 0 for pathological link-cost functions. To see this, assume that $V_\ell = V$ for all ℓ. Then, the bound on efficiency reduces to that in the single link case, i.e.,

$$\mathcal{E}(\{U_m\};\{V_\ell\}) \quad \geq \quad \inf_{c>0} \frac{cv^{-1}(\frac{c}{2}) - V(v^{-1}(\frac{c}{2}))}{cv^{-1}(c) - V(v^{-1}(c))}.$$

Now, applying the arguments in Section V-D, we can identify a sequence of link-cost functions $V^{(n)}$, $n \geq 1$, such that the corresponding sequence of efficiency bounds converges to 0 as $n \to \infty$.

X. Conclusion

This paper was about double auction mechanisms and a proposal for a structured interaction to increase efficiency in the presence of strategic agents. The mechanism has application in data off-loading and network slicing markets. Data offloading is a good low-cost strategy that leverages existing auxiliary technology for handling the growth of mobile data. Technologies to enable such offloading are now available [2], [3]. Network slicing is expected to open up new business opportunities for mobile operators who can slice their physical resources and lease them to tenants or virtual network operators. Since in both examples the resulting markets are resource trading markets, suitable compensation mechanisms have to be put in place to encourage trading of the physical resources. It is natural that the agents involved are strategic. This paper demonstrates that mechanisms for trading resources should be designed with some care. An earlier work proposed a data offloading mechanism (collect bids, allocate offloading amounts, and distribute payments) and designed an iterative procedure to get the system to a competitive equilibrium where all agents benefited, if all agents were price-taking. We showed that if the agents are price-anticipating, this benefit completely disappears and the efficiency loss is 100%. New mechanisms are thus needed when all agents are price-anticipating. We proposed a simple Stackelberg formulation with the supplying agent as a lead player. The resulting mechanism structures the interactions and alleviates the problem to some extent. The efficiency is lower bounded in terms of the true link cost function. The efficiency loss is 25% for quadratic link costs (efficiency = 0.75). While there are link cost functions for which the efficiency loss, even in the Stackelberg formulation, is close to 100%, these appear to be pathological cases. The proposed mechanism with link suppliers as lead players will likely have tolerable efficiency loss for most real link cost functions and arbitrary but linear user payoffs. This is to be contrasted with 100% efficiency loss for the price-anticipating mechanism. Going beyond our scalar bid per resource, our proposal also suggests an interesting open problem for implementation theorists. Does the minimum signaling dimension for social welfare maximization (in the Stackelberg equilibrium solution concept) strictly decrease?

Appendix A
Proof of Theorem 1

The proof is based on Lagrangian technique. We now outline the key steps before going into the details. We first show that the optimality conditions for SYSTEM implies (C1)$-$(C3) that are required of a competitive equilibrium (CE); since the problem SYSTEM is convex, the existence result simply follows from the existence of a (primal and dual optimal) solution to the KKT conditions [15], [16]. For the second part, starting with (C1)$-$(C3) we deduce the optimality conditions for SYSTEM; the second part of the theorem then follows since KKT conditions in our present setting are both necessary and sufficient for optimality. Details follow.

A. Proof of Existence of CE

***Step-1**: KKT conditions for SYSTEM*

The Lagrangian for SYSTEM in (1) is given by

$$\begin{aligned} L(\mathbf{x},\mathbf{y},\lambda,\boldsymbol{\mu}) \quad = \quad & \sum_{m=1}^{M} U_m(x_m) - V\left(\sum_{m=1}^{M} y_m\right) \\ & -\lambda\Big(\sum_m y_m - C\Big) - \sum_m \mu_m \Big(x_m - y_m\Big) \end{aligned}$$

where, λ and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_m)$ are the Lagrange multipliers associated with the constraints in (1b) and (1c), respectively. Defining $y = \sum_i y_i$, the optimality conditions are given by

$$\left.\begin{array}{ll} U'_m(x_m) = \mu_m & \text{if } x_m > 0 \\ U'_m(0) \leq \mu_m & \text{if } x_m = 0 \end{array}\right\} \tag{47}$$

$$\left.\begin{array}{ll} V'(y) + \lambda = \mu_m & \text{if } y_m > 0 \\ V'(y) + \lambda \geq \mu_m & \text{if } y_m = 0 \end{array}\right\} \tag{48}$$

along with primal feasibility ((1b)$-$(1d)), dual feasibility (i.e., $\lambda \geq 0$, $\mu_m \geq 0\ \forall m$) and complementary slackness conditions:

$$\lambda\Big(\sum_m y_m - C\Big) \quad = \quad 0 \tag{49}$$

$$\mu_m\Big(x_m - y_m\Big) \quad = \quad 0\ \forall m. \tag{50}$$

***Step-2**: Identifying a candidate competitive-equilibrium*

Since the problem is convex there exist primal and dual optimal points, $(\mathbf{x}^s, \mathbf{y}^s)$ and $(\lambda^s, \boldsymbol{\mu}^s)$, respectively, that together satisfy the above KKT conditions. Define $\mathbf{p}^s$ and $\boldsymbol{\beta}^s$ as

$$p^s_m \quad = \quad x^s_m \mu^s_m \tag{51}$$

$$\beta^s_m \quad = \quad \begin{cases} \frac{y^s_m}{(\mu^s_m - \lambda^s)} & \text{if } \mu^s_m \neq \lambda_s \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

In the following we will show that $(\mathbf{p}^s, \boldsymbol{\beta}^s, \lambda^s, \boldsymbol{\mu}^s)$ is a competitive equilibrium, i.e., we verify (C1)$-$(C3).

***Step-3**: Verifying (C1) and (C2)*

To verify (C1), we need to show that p^s_m is optimal for the problem of maximizing $P_m(p_m;\mu^s_m)$ in (3) over $p_m \geq 0$, the optimality condition for which is given by

$$\begin{array}{ll} U'_m\left(\frac{p_m}{\mu^s_m}\right) = \mu^s_m & \text{if } p_m > 0 \\ U'_m(0) \leq \mu^s_m & \text{if } p_m = 0. \end{array}$$

From (47) and (51) it follows that p_m^s satisfies the above condition, thus implying (C1).

Similarly, the optimality condition for maximizing the link pay-off function, $P_L\Big(\boldsymbol{\beta};(\boldsymbol{\mu}^s,\lambda^s)\Big)$ in (4) over $\boldsymbol{\beta}\geq\mathbf{0}$ is:

- if $\beta_m>0$

$$V'\left(\sum_i\beta_i\Big(\mu_i^s-\lambda^s\Big)\right)\Big(\mu_m^s-\lambda^s\Big)=\Big(\mu_m^s-\lambda^s\Big)^2; \quad (53)$$

- if $\beta_m=0$

$$V'\left(\sum_i\beta_i\Big(\mu_i^s-\lambda^s\Big)\right)\Big(\mu_m^s-\lambda^s\Big)\geq\Big(\mu_m^s-\lambda^s\Big)^2. \quad (54)$$

We show that β_m^s in (52) satisfies the above. Suppose $\beta_m^s>0$ then it should be that $\mu_m^s\neq\lambda^s$. Thus, with $\boldsymbol{\beta}=\boldsymbol{\beta}^s$, expression (53) simplifies to

$$V'\left(\sum_i\beta_i^s\Big(\mu_i^s-\lambda^s\Big)\right)+\lambda^s=\mu_m^s=:\mu^s \quad (55)$$

which follows from the first part of (48). On the other hand, if $\beta_m^s=0$ then either $\mu_m^s=\lambda_s$ or $y_m^s=0$ (or both). If the former is true then (54) holds trivially, while in the latter case (54) follows from the second part of (48). Thus, condition (C2) is verified.

***Step-4**: Verifying (C3)*

To verify (C3-a) we need to prove that $\mathbf{x}^s=\mathbf{y}^s$. Evidently, since the objective in (1a) is strictly increasing in x_m (owing to the strictly-increasing condition imposed on U_m), at optimality it should be that $\mathbf{x}^s=\mathbf{y}^s$. For completeness, we formally prove this result using the optimality conditions. Suppose $0\leq x_m^s<y_m^s$ for some m. Then, the slackness condition in (50) immediately implies that $\mu_m^s=0$, so that from (48) we obtain $V'(y^s)+\lambda^s=0$. Since V is strictly increasing and $y_m^s>0$, we have $V'(y^s)>0$ yielding $\lambda^s<0$ which is a contradiction. Thus, $x_m^s=y_m^s\ \forall m$.

Since (C2) is already verified, from (55) we see that we have $\mu_m^s=\mu^s$ for all $m\in\mathcal{M}^s$, where $\mathcal{M}^s=\{m:\mu_m^s\neq\lambda^s\}$. Thus, (summing the expression in (51) over all m) we have

$$\begin{aligned}\sum_m x_m^s=\sum_m\frac{p_m^s}{\mu_m^s}&=\sum_{m\in\mathcal{M}^s}\frac{p_m^s}{\mu^s}+\sum_{m\notin\mathcal{M}^s}\frac{p_m^s}{\mu_m^s}\\&=\frac{1}{\mu^s}\sum_m p_m^s. \quad (56)\end{aligned}$$

The last equality is obtained by noting that $p_s^m=0$ for $m\notin\mathcal{M}^s$. This is because, when $\mu_m^s=\lambda^s$, from (48) we see that $y_m^s=0$ (as $y_m^s>0$ leads to the contradiction $V'(y^s)=0$); now, since $x_m^s=y_m^s$ (from (C3-a)) we obtain $p_m^s=0$ from (51). This establishes (56).

Similarly, using (52) and by splitting the following sum over $m\in\mathcal{M}^s$ and $m\notin\mathcal{M}^s$ separately, we obtain

$$\sum_m y_m^s=\sum_m\beta_m^s\Big(\mu_m^s-\lambda^s\Big)=\Big(\mu^s-\lambda^s\Big)\sum_{m\in\mathcal{M}^s}\beta_m^s. \quad (57)$$

Note that, unlike in (56), the summation in the final term above is restricted to $m\in\mathcal{M}^s$. This is because, in this case whenever $\mu_m^s\neq\lambda^s$ it is not necessary that $\beta_m^s=0$. We need to consider two cases:

- Suppose $\lambda^s=0$. Then, since $\sum_m x_m=\sum_m y_m$ (recall (C3-a)), solving for μ^s from (56) and (57) we obtain
$$\mu^s=\sum_m p_m^s/\widehat{C}^s$$
where $\widehat{C}^s$ is as in (5) (but with $(\mathbf{p},\boldsymbol{\beta})$ replaced with $(\mathbf{p}^s,\boldsymbol{\beta}^s)$). Using the above in (56) yields $\sum_m x_m^s=\widehat{C}$. Since $\sum_m x_m^s\leq C$, we have $\widehat{C}^s\leq C$ under this case.
- Suppose $\lambda^s>0$. Then the slackness condition in (49) immediately implies $\sum_m y_m=C=\sum_m x_m$. Thus, from (56), we have
$$\mu^s=\sum_m p_m^s/C.$$
Using this in (57) and simplifying for λ^s we obtain
$$\lambda^s=\left(1-\left(\frac{C}{\widehat{C}^s}\right)^2\right)\frac{\sum_i p_i^s}{C}.$$
Note that, since $\lambda^s>0$, we have $\widehat{C}^s>C$ in this case.

Results from the above two conditions can be compactly expressed as (7) and (8); thus (C3-b) and (C3-c) are verified.

B. *Proof of Optimality of CE*

Let $(\mathbf{p}^c,\boldsymbol{\beta}^c,\lambda^c,\boldsymbol{\mu}^c)$ be a competitive equilibrium. Define rate-vectors $\mathbf{x}^c$ and $\mathbf{y}^c$ as $x_m^c=\frac{p_m^c}{\mu_m^c}$ and $y_m^c=\beta_m^c(\mu_m^c-\lambda^c)$. We will show that $(x^c,y^c,\lambda^c,\mu^c)$ satisfies the KKT conditions for the problem SYSTEM (recall *Step-1*).

From (C1) it follows that p_m^c is optimal for the user problem of maximizing the pay-off function $P_m(p_m;\mu_m^c)$ in (3) over all $p_m\geq 0$. Thus we have

$$\begin{array}{ll}U_m'\left(\frac{p_m^c}{\mu_m^c}\right)=\mu_m^c & \text{if } p_m^c>0\\ U_m'(0)\leq\mu_m^c & \text{if } p_m^c=0.\end{array}$$

With $x_m^c:=\frac{p_m^c}{\mu_m^c}$ the above expression is identical to the KKT condition in (47).

Similarly, (C2) implies that $\boldsymbol{\beta}^c$ is optimal for the link's problem of maximizing $P_L\Big(\boldsymbol{\beta};(\boldsymbol{\mu}^c,\lambda^c)\Big)$ in (4) over $\boldsymbol{\beta}\geq\mathbf{0}$. Thus, $\boldsymbol{\beta}^c$ satisfies the optimality conditions in (53) and (54), but with $(\lambda^s,\boldsymbol{\mu}^s)$ replaced by $(\lambda^c,\boldsymbol{\mu}^c)$.

- Suppose $y_m^c=\beta_m^c(\mu_m^c-\lambda^c)>0$ then $\beta_m^c>0$ and $\mu_m^c\neq\lambda^c$; the optimality condition in this case simplifies to the expression in (55) (but again with $(\boldsymbol{\beta}^s,\lambda^s,\boldsymbol{\mu}^s)$ replaced by $(\boldsymbol{\beta}^c,\lambda^c,\boldsymbol{\mu}^c)$). Then, substituting $y_m^c=\beta_m^c(\mu_m^c-\lambda^c)$, we obtain the first part of the KKT condition in (48) is satisfied.
- Suppose $y_m^c=0$ then either $\mu_m^c=\lambda^c$ or $\beta_m^c=0$ (or both). $\mu_m^c=\lambda^c$ case trivially satisfies the optimality conditions in (53) and (54). If $\beta_m^c=0$ then the second part of (48) follows from (54).

For the slackness conditions in (49) and (50), note that condition (C3-1) in (6) implies $x_m^c=y_m^c$ $(\forall m)$ so that (50) holds immediately. To show (49), first suppose that $\lambda^c>0$.

Then, from (C3-c) in (8) we have $\widehat{C}^c > C$ (where $\widehat{C}^c$ is as in (5) but with $(\mathbf{p}, \boldsymbol{\beta})$ replaced by $(\mathbf{p}^c, \boldsymbol{\beta}^c)$). Using the above condition in (7) of (C3-b) we obtain $\mu = \sum_i p_i^c / C$. Thus

$$\begin{aligned} \sum_m y_m^c = \sum_m x_m^c &= \sum_{m \in \mathcal{M}} \frac{p_m^c}{\mu_m^c} + \sum_{m \notin \mathcal{M}} \frac{p_m^c}{\mu_m^c} \\ &= \frac{\sum_m p_m^c}{\mu} = C, \end{aligned}$$

where we have used the fact that $p_m^c = 0$ for $m \notin \mathcal{M}$ to get the last equality. Next, suppose that $\sum_m y_m^c < C$. In this case we have

$$C > \sum_m y_m^c = \sum_m x_m^c = \sum_m p_m^c / \mu = \min\left\{C, \widehat{C}^c\right\}$$

implying $\widehat{C}^c < C$ so that from (C3-c) in (8) we obtain $\lambda^c = 0$. Thus, the slackness condition in (49) is verified.

Finally, since the problem SYSTEM in (1) is convex, the sufficiency of the KKT conditions [15], [16] imply that $(\mathbf{x}^c, \mathbf{y}^c)$ is (primal) optimal. ■

APPENDIX B
PROOF OF RESULTS IN SECTION IV

A. Proof of Lemma 1

The Lagrangian for the problem NETWORK in (2) is given by

$$\begin{aligned} L(\mathbf{x}, \mathbf{y}, \lambda, \boldsymbol{\mu}) &= \sum_m p_m \log(x_m) - \sum_m \frac{y_m^2}{2\beta_m} \\ &\quad -\lambda\Big(\sum_m y_m - C\Big) - \sum_m \mu_m \Big(x_m - y_m\Big) \end{aligned}$$

where λ and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_m)$ are the Lagrange multipliers associated with the constraints in (2b) and (2c), respectively. The optimality (KKT) conditions include

$$\frac{\partial L}{\partial x_m} = \frac{p_m}{x_m} - \mu_m = 0 \ \forall m \tag{58}$$

$$\frac{\partial L}{\partial y_m} = -\frac{y_m}{\beta_m} - \lambda + \mu_m = 0 \ \forall m \tag{59}$$

along with primal feasibility ((2b), (2c) and (2d)), dual feasiblity ($\lambda \geq 0$ and $\boldsymbol{\mu} \geq \mathbf{0}$), and the following complementary slackness conditions:

$$\lambda\Big(\sum_m y_m - C\Big) = 0 \tag{60}$$

$$\mu_m \Big(x_m - y_m\Big) = 0 \ \forall m. \tag{61}$$

Note that the objective in (2a) is strictly increasing with x_m; hence, at optimality it should be that $x_m = y_m$ for all m. As in the proof of Theorem 1 (see *Step-4*), this observation can also be argued formally using the KKT conditions. Suppose at optimality we have $x_m < y_m$ for some m. Then, the slackness condition in (61) implies $\mu_m = 0$, using which in (59) we obtain $y_m = -\beta_m \lambda$. However, non-negativity constraint on all the variables forces $\beta_m \lambda = 0$, yielding $0 \leq x_m = y_m = 0$, which is a contradiction. Thus, at optimality we must have

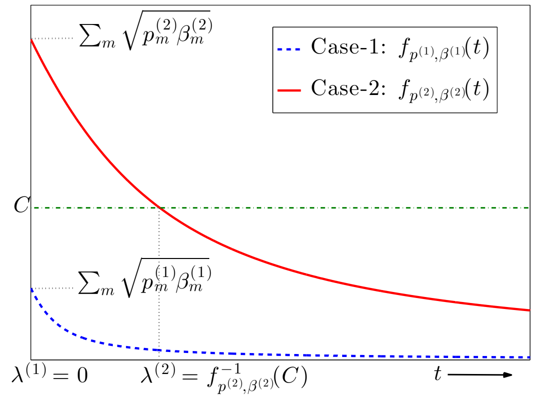

Fig. 2. Illustration of the two cases that are possible depending on whether $f_{\mathbf{p},\boldsymbol{\beta}}(0) = \sum_m \sqrt{p_m \beta_m} \leq C$ (*Case-1*) or otherwise (*Case-2*). In the above depiction $(\mathbf{p}^{(1)}, \boldsymbol{\beta}^{(1)})$ and $(\mathbf{p}^{(2)}, \boldsymbol{\beta}^{(2)})$ are such that they satisfy *case-1* and *case-2*, respectively, so that $\lambda^{(1)} := \lambda(\mathbf{p}^{(1)}, \boldsymbol{\beta}^{(1)}) = 0$ and $\lambda^{(2)} := \lambda(\mathbf{p}^{(2)}, \boldsymbol{\beta}^{(2)}) = f^{-1}_{\mathbf{p}^{(2)},\boldsymbol{\beta}^{(2)}}(C)$.

$x_m = y_m \ \forall m$. Now, using (58) and (59) in the above expression we have

$$x_m = \frac{p_m}{\mu_m} = \beta_m \Big(\mu_m - \lambda\Big) = y_m \ \forall m. \tag{62}$$

Solving this for μ_m we obtain (as required; recall (11))

$$\mu_m = \frac{\lambda + \sqrt{\lambda^2 + 4\frac{p_m}{\beta_m}}}{2} \ \forall m. \tag{63}$$

We next proceed to obtain λ. For this, using the above expression in (62) and summing over all m, we obtain

$$\sum_m x_m = f_{\mathbf{p},\boldsymbol{\beta}}(\lambda) = \sum_m y_m \leq C, \tag{64}$$

where the function $f_{\mathbf{p},\boldsymbol{\beta}}$ is as in (10), and the inequality is simply due to the capacity constraint in (2b). Note that, $f_{\mathbf{p},\boldsymbol{\beta}}(\lambda)$ as a function of λ is strictly decreasing with $f_{\mathbf{p},\boldsymbol{\beta}}(0) = \sum_i \sqrt{p_i \beta_i}$ (see Fig. 2 for an illustration). Also, $\lim_{\lambda \to \infty} f_{\mathbf{p},\boldsymbol{\beta}}(\lambda) = 0$. Two cases are possible (recall (9)) as follows.

Case-1 ($f_{\mathbf{p},\boldsymbol{\beta}}(0) \leq C$): In this case $\lambda = 0$ alone satisfies (64) while ensuring the slackness condition in (60).

Case-2 ($f_{\mathbf{p},\boldsymbol{\beta}}(0) > C$): In this case we require $\lambda > 0$, since $\lambda = 0$ cannot satisfy (64). However, $\lambda > 0$ immediately implies $\sum_m y_m = C$ (see (60)). Hence, we set $\lambda = f^{-1}_{\mathbf{p},\boldsymbol{\beta}}(C)$. See Fig. 2 for an illustration of the above two cases. ■

B. Proof of Theorem 2

We will first show that $(\mathbf{p}^o, \boldsymbol{\beta}^o)$ is a Nash equilibrium. For this, note that once the link-supplier fixes his bids to $\boldsymbol{\beta}^o = \mathbf{0}$, then for any vector of user bids $\mathbf{p} \geq \mathbf{0}$ the system operates in the regime $\sum_i \sqrt{p_i \beta_i^o} \leq C$. Thus, using the first expression in (12), for any $\overline{p}_m > 0$, we have

$$\begin{aligned} Q_m(\overline{p}_m, \mathbf{p}^o_{-m}, \boldsymbol{\beta}^o) &= U_m(0) - \overline{p}_m \\ &< U_m(0) \end{aligned}$$

$$= \quad Q_m(p_m^o, \mathbf{p}_{-m}^o, \boldsymbol{\beta}^o).$$

Thus, unilateral deviation from p_m^o is not beneficial for user m ($\forall m$). Similarly, for any $\overline{\boldsymbol{\beta}}$ such that $\overline{\beta}_m > 0$ for some m, we have

$$Q_L(\overline{\boldsymbol{\beta}}, \mathbf{p}^o) = -V(0) + \sum_m p_m^o = Q_L(\boldsymbol{\beta}^o, \mathbf{p}^o)$$

To obtain the above, note that since the users' payments are zero, from (13), the first expression applies. Any other value of β_m does not strictly increase the pay-off of the link-supplier. Thus, $(\mathbf{p}^o, \boldsymbol{\beta}^o)$ is a Nash equilibrium.

We now prove the uniqueness of the Nash equilibrium. Let $(\mathbf{p}^*, \boldsymbol{\beta}^*)$ be a Nash equilibrium. Suppose $p_m^* > 0$ for some m. Then, if $\beta_m^* = 0$ (recalling (12)) the pay-off to user m is

$$\begin{aligned} Q_m(p_m^*, \mathbf{p}_{-\mathbf{m}}^*, \boldsymbol{\beta}^*) &= U_m(0) - p_m^* < U_m(0) \\ &= Q_m(0, \mathbf{p}_{-\mathbf{m}}^*, \boldsymbol{\beta}^*) \end{aligned}$$

which contradicts the assumption that $(\mathbf{p}^*, \boldsymbol{\beta}^*)$ is a Nash equilibrium. On the other hand, if $\beta_m^* > 0$, then the link-supplier can benefit by deviating to the bid $\boldsymbol{\beta}^o$. This is because, the rate-cost incurred by deviating to $\boldsymbol{\beta}^o$ is always strictly lower (since he now provides zero bandwidth). Also, the payment $\sum_m p_m^*$ accrued under $\beta_m^* > 0$ may be better if the system was not already in the regime $\sum_i \sqrt{p_i^* \beta_i^*} \leq C$; if already in that regime the payment remains unchanged. Formally,

$$Q_L(\boldsymbol{\beta}^*, \mathbf{p}^*) < -V(0) + \sum_m p_m^* = Q_L(\boldsymbol{\beta}^o, \mathbf{p}^*)$$

which is again a contradiction. Thus, $p_m^* = 0 \; \forall m$, i.e., $\mathbf{p}^* = \mathbf{0}$.

Now, suppose $\beta_m^* > 0$ for some m. User m can benefit by making a small payment $\overline{p}_m$. Indeed choose a $\overline{p}_m$ satisfying

$$0 < \overline{p}_m \leq \min\left\{C^2/\beta_m^*, q_m\right\} \tag{65}$$

where q_m is the maximizer of the function

$$h(p_m) \quad = \quad U_m\left(\sqrt{p_m \beta_m^*}\right) - p_m \tag{66}$$

over $p_m \geq 0$. Note that $h(p_m)$ is strictly concave in p_m. Hence, q_m is the unique solution to the optimality condition

$$U_m'\left(\sqrt{q_m \beta_m^*}\right) = 2\sqrt{q_m/\beta_m^*}.$$

Since $U'(\cdot)$ is strictly decreasing with $U'(0) > 0$, we have $q_m > 0$, thus enabling us to choose a $\overline{p}_m$ satisfying (65). The $\min$ term in (65) involving C^2/β_m^* is required to ensure that the first expression of (12) is applicable. Thus, we have

$$\begin{aligned} Q_m(p_m^*, \mathbf{p}_{-\mathbf{m}}^*, \boldsymbol{\beta}^*) &= U_m(0) \text{ (since } p_m^* = 0) \\ &< U_m\left(\sqrt{\overline{p}_m \beta_m^*}\right) - \overline{p}_m \\ &= Q_m(\overline{p}_m, \mathbf{p}_{-\mathbf{m}}^*, \boldsymbol{\beta}^*) \end{aligned}$$

where the inequality is because the function $h(\cdot)$, being strictly concave, is strictly increasing until q_m. The above contradiction implies that $\beta_m^* = 0 \; \forall m$, i.e., $\boldsymbol{\beta}^* = \mathbf{0}$. Hence, $(\mathbf{p}^o, \boldsymbol{\beta}^o)$ is the only Nash equilibrium. ■

APPENDIX C
PROOF OF THEOREM 3

We first show that the objective function of the problem in (18) is continuous in β_m for each m. We next argue that it is sufficient to consider $\arg\max$ in (18) over a compact set of $\boldsymbol{\beta}$ values. Then the proof is completed by invoking Weierstrass theorem. The details are as follows.

Proof of continuity: To show the continuity of the objective in (18) it suffices to prove that r_{β_m} is continuous in β_m (for all m). For simplicity we omit the subscript m hereafter. Let us first prove right-continuity of r_β at $\beta = 0$. Clearly, since r_β is the solution to $U_m'(r) = 2r/\beta$, we see that $r_\beta \geq 0$ and, moreover, since U_m' is strictly decreasing we have $r_\beta = \frac{U_m'(r_\beta)}{2}\beta \leq \frac{U_m'(0)}{2}\beta$ from which it follows that $r_0 = 0$ so that r_β is continuous at $\beta = 0$. To prove continuity at any $\beta > 0$, we show that r_β is Lipschitz continuous, i.e., $|r_{\beta_1} - r_{\beta_2}| \leq \frac{U_m'(0)}{2}|\beta_1 - \beta_2|$. Without loss of generality assume $\beta_2 > \beta_1$ so that $r_{\beta_2} > r_{\beta_1}$. Then we have (again since $U_m'(\cdot)$ is decreasing)

$$\begin{aligned} r_{\beta_2} - r_{\beta_1} &= U_m'(r_{\beta_2})\beta_2/2 - U_m'(r_{\beta_1})\beta_1/2 \\ &\leq U_m'(r_{\beta_2})\beta_2/2 - U_m'(r_{\beta_2})\beta_1/2 \\ &= (U_m'(r_{\beta_2})/2)(\beta_2 - \beta_1) \\ &\leq (U_m'(0)/2)(\beta_2 - \beta_1). \end{aligned}$$

This establishes Lipschitz continuity.

Proof that it suffices to search for β_m in a bounded set: First note that, using the definition of r_{β_m} and p_m^β, we may write the objective function as

$$\begin{aligned} & -V\left(\sum_m \frac{p_m^{\boldsymbol{\beta}}}{r_{\beta_m}/\beta_m}\right) + \sum_m p_m^{\boldsymbol{\beta}} \\ &= -V\left(\sum_m \frac{p_m^{\boldsymbol{\beta}}}{U_m'(r_{\beta_m})/2}\right) + \sum_m p_m^{\boldsymbol{\beta}} \\ &\leq -V\left(\sum_m \frac{p_m^{\boldsymbol{\beta}}}{U_m'(0)/2}\right) + \sum_m p_m^{\boldsymbol{\beta}} \end{aligned}$$

since $V(\cdot)$ is strictly increasing. From the assumption that $V(x)/x \to \infty$ as $x \to \infty$, we see that the last term in the right-hand side above is less than 0 for all $(p_m^\beta, 1 \leq m \leq M)$ with $\sum_m p_m^\beta > P$ for some bounded P. Since each $p_m^\beta \geq 0$, we trivially have that $0 \leq p_m^\beta \leq P$.

From the formula for r_{β_m} and p_m^β, we have $p_m^\beta = r_{\beta_m} U_m'(r_{\beta_m})/2$, and hence $0 \leq r_{\beta_m} U_m'(r_{\beta_m})/2 \leq P$. Under the assumption $rU_m'(r) \to \infty$ as $r \to \infty$, we must then have $0 \leq r_{\beta_m} \leq R$ for some bounded R, and since U_m' is strictly decreasing and strictly positive, we must have $U_m'(r_{\beta_m}) \geq U_m'(R) > 0$. Using this, we then have

$$\beta_m = p_m^\beta/(U_m'(r_{\beta_m})/2)^2 \leq 4P/(U_m'(R))^2 < \infty.$$

This completes the proof. ■

APPENDIX D
PROOF OF THEOREM 4

Consider linear pay-offs of the form $U_m(x_m) = c_m x_m$ where $c_m > 0$ ($m = 1, 2, \cdots, M$). Without loss of generality, assume that $c_1 = \max_m\{c_m\}$. Then, recalling (22), the utility at Stackelberg equilibrium can be written as

$$\begin{aligned}
&\text{Stackelberg utility} \\
&= \sum_m U_m\left(x_m^{\boldsymbol{\beta}^*}\right) - V\left(\sum_m x_m^{\boldsymbol{\beta}^*}\right) \\
&= U_1\left(v^{-1}\left(\frac{c_1}{2}\right)\right) - V\left(v^{-1}\left(\frac{c_1}{2}\right)\right) \\
&= c_1 v^{-1}\left(\frac{c_1}{2}\right) - V\left(v^{-1}\left(\frac{c_1}{2}\right)\right). \qquad (67)
\end{aligned}$$

Next, the social optimal utility is obtained by solving

$$\max_{\mathbf{x}\geq\mathbf{0}}\left\{\sum_m U_m(x_m) - V\left(\sum_m x_m\right)\right\}.$$

Substituting for the linear pay-off functions and rearranging, the above problem can be alternatively expressed as

$$\max_{\mathbf{X}\geq\mathbf{0}}\left\{\sum_m c_m x_m - V\left(\sum_m x_m\right)\right\}.$$

Thus, the optimal rate allocation x_m^S is given by

$$x_m^S = \begin{cases} v^{-1}(c_1) & \text{if } m = 1 \\ 0 & \text{otherwise.}\end{cases}$$

Thus, the social optimal utility is given by

$$\text{Social utility} = \sum_\ell \left(c_1 v^{-1}(c_1) - V\left(v^{-1}(c_1)\right)\right). \qquad (68)$$

From (67) and (68) we have (replacing c_1 by c)

$$\mathcal{E}(\{U_m\}; V) \geq \frac{cv^{-1}(\frac{c}{2}) - V(v^{-1}(\frac{c}{2}))}{cv^{-1}(c) - V(v^{-1}(c))}.$$

The worst case bound on efficiency can be obtained by taking infimum over all $c > 0$. ■

Appendix E
Proof of Theorem 6

The proof of the first part, that $(\mathbf{P}^o, \mathbf{A}^o)$ is an ML Nash equilibrium, is identical to the proof of the corresponding part in Theorem 2. The uniqueness part however requires some modifications since the user pay-offs are now functions of its payments-to and signals-from all the links.

As in the proof of Theorem 2, we begin by assuming that $(\mathbf{P}^*, \mathbf{B}^*)$ is an ML Nash equilibrium. Suppose that $p_{m\ell}^* > 0$ for some (m, ℓ). For simplicity define

$$\delta_{m\ell} = \begin{cases} \sqrt{p_{m\ell}^*\beta_{m\ell}^*} & \text{if } \sum_i \sqrt{p_{i\ell}^*\beta_{i\ell}^*} \leq C_\ell \\ \frac{2p_{m\ell}^*}{\lambda_\ell^* + \sqrt{\lambda_\ell^{*2} + 4\frac{p_{m\ell}^*}{\beta_{m\ell}^*}}} & \text{otherwise}\end{cases}$$

where $\lambda_\ell^* = \lambda_\ell(\mathbf{P}^*, \mathbf{B}^*)$. If $\beta_{m\ell}^* = 0$, then $\delta_{m\ell} = 0$ and we can write

$$\begin{aligned}
Q_m(\mathbf{p}_m^*, \mathbf{p}_{-\mathbf{m}}^*, \mathbf{B}^*) &= U_m\left(\sum_j \delta_{mj}\right) - \sum_j p_{mj}^* \\
&= U_m\left(\sum_{j\neq\ell} \delta_{mj}\right) - \sum_j p_{mj}^* \\
&< U_m\left(\sum_{j\neq\ell} \delta_{mj}\right) - \sum_{j\neq\ell} p_{mj}^* \\
&= Q_m(\overline{\mathbf{p}}_m, \mathbf{p}_{-\mathbf{m}}^*, \mathbf{B}^*)
\end{aligned}$$

where $\overline{\mathbf{p}}_m$ is such that $\overline{p}_{m\ell} = 0$ while $\overline{p}_{mj} = p_{mj}^*\ \forall j \neq \ell$. The above is a contradiction to our assumption that $(\mathbf{P}^*, \mathbf{B}^*)$ is an ML Nash equilibrium. On the other hand, if $\beta_{m\ell}^* > 0$, we have

$$\begin{aligned}
Q_{L,\ell}(\boldsymbol{\beta}_\ell^*, \boldsymbol{\beta}_{-\boldsymbol{\ell}}^*, \mathbf{P}^*) &< -V_\ell(0) + \sum_m p_{m\ell}^* \\
&= Q_{L,\ell}(\boldsymbol{\beta}_\ell^o, \boldsymbol{\beta}_{-\boldsymbol{\ell}}^*, \mathbf{P}^*)
\end{aligned}$$

a contradiction. Hence, $p_{m\ell}^* = 0\ \forall(m, \ell)$, i.e., $\mathbf{P}^* = \mathbf{0}$.

Now, suppose $\beta_{m\ell}^* > 0$ for some (m, ℓ). Then, as in the proof of Theorem 2, we have

$$\begin{aligned}
Q_m(\mathbf{p}_m^*, \mathbf{p}_{-\mathbf{m}}^*, \mathbf{B}^*) &= U_m(0) \ (\text{since } \mathbf{p}_m^* = \mathbf{0}) \\
&< U_m\left(\sqrt{\overline{p}_{m\ell}\beta_{m\ell}^*}\right) - \overline{p}_{m\ell} \\
&= Q_m(\overline{\mathbf{p}}_m, \mathbf{p}_{-\mathbf{m}}^*, \mathbf{B}^*)
\end{aligned}$$

where $\overline{\mathbf{p}}_m$ is such that $\overline{p}_{mj} = p_{mj}^*\ \forall \ell \neq j$ and $\overline{p}_{m\ell}$ satisfies

$$0 < \overline{p}_{m\ell} \leq \min\left\{C_\ell^2/\beta_{m\ell}^*, q_m\right\}$$

where q_m is the maximizer of the function $h(\cdot)$ in (66). Thus, $\beta_{m\ell}^* = 0\ \forall(m, \ell)$ (i.e., $\mathbf{B}^* = \mathbf{0}$) for a Nash equilibrium. This completes the proof. ■

Appendix F
Proof of Theorem 7

The proof is along the lines of the proof of Theorem 4. For completeness we however present the details again here.

Consider linear pay-offs of the form $U_m(x_m) = c_m x_m$ where $c_m > 0$ $(m = 1, 2, \cdots, M)$. Without loss of generality, assume that $c_1 = \max_m\{c_m\}$. Then, recalling (42) and (44), the utility at Stackelberg equilibrium can be written as

$$\begin{aligned}
&\text{Stackelberg utility} \\
&= \sum_m U_m\left(\sum_\ell x_{m\ell}^{\mathbf{B}^*}\right) - \sum_\ell V_\ell\left(\sum_m x_{m\ell}^{\mathbf{B}^*}\right) \\
&= U_1\left(\sum_\ell v_\ell^{-1}\left(\frac{c_1}{2}\right)\right) - \sum_\ell V_\ell\left(v_\ell^{-1}\left(\frac{c_1}{2}\right)\right) \\
&= c_1\sum_\ell v_\ell^{-1}\left(\frac{c_1}{2}\right) - \sum_\ell V_\ell\left(v_\ell^{-1}\left(\frac{c_1}{2}\right)\right) \\
&= \sum_\ell\left(c_1 v_\ell^{-1}\left(\frac{c_1}{2}\right) - V_\ell\left(v_\ell^{-1}\left(\frac{c_1}{2}\right)\right)\right). \qquad (69)
\end{aligned}$$

Next, the social optimal utility is obtained by solving

$$\max_{\mathbf{X}\geq\mathbf{0}}\left\{\sum_m U_m\left(\sum_\ell x_{m\ell}\right) - \sum_\ell V_\ell\left(\sum_m y_{m\ell}\right)\right\}.$$

Substituting for the linear pay-off functions and rearranging, the above problem can be alternatively expressed as

$$\max_{\mathbf{X}\geq\mathbf{0}}\left\{\sum_\ell\left(\sum_m c_m x_{m\ell} - V_\ell\left(\sum_m x_{m\ell}\right)\right)\right\}.$$

Thus, for each ℓ the optimal rate allocation $x^S_{m\ell}$ is given by

$$x^S_{m\ell} = \begin{cases} v_\ell^{-1}(a_1) & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the social optimal utility is given by

$$\text{Social utility} = \sum_\ell \left(c_1 v_\ell^{-1}(c_1) - V_\ell\left(v_\ell^{-1}(c_1)\right)\right). \quad (70)$$

From (69) and (70) we have (replacing c_1 by c)

$$\mathcal{E}(\{U_m\};\{V_\ell\}) \geq \frac{\sum_\ell \left(c v_\ell^{-1}(\frac{c}{2}) - V_\ell(v_\ell^{-1}(\frac{c}{2}))\right)}{\sum_\ell \left(c v_\ell^{-1}(c) - V_\ell(v_\ell^{-1}(c))\right)}.$$

The worst case bound on efficiency can be obtained by taking infimum over all $c > 0$. ■

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