Observability of Discrete-Time LTI Systems under Unknown Piece-wise Constant Inputs

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Abstract— **This paper is on observability of discrete-time LTI systems under unknown piece-wise constant inputs with sufficiently slow, but arbitrary update times. Assuming knowledge of the update times, we characterize the unobservable subspace and show that with sufficiently many measurements in each inter-update interval of the input, the unobservable subspace remains fixed. We explore the implications of the result for privacy in event-triggered control through an illustrative example.**

I. Introduction

Observability under unknown inputs has been a topic of interest to the controls community for several decades. Motivated by the recent research trend of event-triggered control, we revisit the classical problem of observability.

Literature Review:

The literature on observability of linear time invariant (LTI) systems under unknown or partially known inputs stretches back to late 1960s. Some early works on the topic are [1]–[4]. More recent works on the topic include sliding mode observer for unknown input and state estimation [5], observability under unknown inputs in the context of singular differential algebraic systems [6], structural input and state observability [7], time-delayed observers [8], [9] and in the context of switched systems [10], [11].

It is well known that if a continuous-time LTI system is observable under known inputs then periodic sampling retains that property except for some pathological sampling periods [12]. The increasing popularity of event-triggered control [13]–[16] raises the question of observability under aperiodic sampling, a topic on which there is currently very limited work [17]. Further, to the best of our knowledge, there is no existing work on observability under an aperiodically updated, unknown piece-wise constant control.

The topic of this paper is also relevant for privacy in event-triggered control. While there exist some papers on privacy preserving or secure event-triggered control, such as [18]–[21], there is no work that studies the privacy implications of existing event-triggered controllers. Such a study is particularly important given that in event-triggered control there is implicit information in the event times about the state of the system [22].

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Contributions: In this paper, we characterize the unobservable subspace of a discrete-time LTI system under unknown piece-wise constant input but known, possibly aperiodic, update times of the input. First, we study the problem under the assumption of a constant unknown input and then extend the ideas to the case of piece-wise constant unknown input. In particular, we show that if the updates to the input are slow enough then with sufficiently many measurements in each inter-update interval of the input, the unobservable subspace remains fixed with time. We apply the result to a system with event-triggered control in the presence of an eavesdropper that has access to the sensor measurements, the control input update times and the triggering rule. We demonstrate, through an example, that if the triggering rule is event based then the state can be identified up to a bounded set, whose "size" decreases with time to zero. On the other hand, for time-triggered updates of the input, the eavesdropper can infer nothing about the component of the state in the unobservable subspace.

Notation: We let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}_0$ be the set of real numbers, integers and the set of natural numbers including zero, respectively. For $x \in \mathbb{R}^n$, we let $||x||$ be the Euclidean norm of x. We use the notation $[K_i, K_{i+1})_{\mathbb{Z}}$ for $[K_i, K_{i+1}) \cap \mathbb{Z}$. We use similar notation for closed, open and the other half-open intervals. We use $\mathbf{0}$ to denote the zero matrix of appropriate dimensions. For vectors v and w, we use (v, w) to represent the vector $[v^T,\ w^T]^T$. We use $M^\dagger$, $\mathrm{Ker}(M)$ and $\mathrm{Im}\,(M)$ for the pseudo-inverse, null-space and the image space of the matrix M, respectively. We let $\mathcal{Z}^\perp$ be the orthogonal complement of a subspace $\mathcal{Z}$. We use I_r to represent the identity matrix of dimension r.

II. Problem Formulation

We consider a discrete-time linear time-invariant system

$$x(k+1) = Ax(k) + Bu(k) \tag{1a}$$
$$y(k) = Cx(k) \tag{1b}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ are the plant state, the control input and the measurement, respectively. We assume that the input is piece-wise constant i.e.

$$u(k) = u(K_i),\ \forall k \in [K_i, K_{i+1})_{\mathbb{Z}}, \tag{1c}$$

where $\mathcal{K} := \{K_i\}_{i \in \mathbb{N}_0}$ is the increasing sequence of input *update times*. We call the set of time steps $[K_i, K_{i+1})_{\mathbb{Z}}$ as the i^{th} *inter-update interval*.

A. Assumptions

We make the following assumptions about the system (1) and the input (1c).

A1 No eigenvalue of matrix A is equal to 1.

A2 Matrices B and C are full column rank and full row rank matrices, respectively.

A3 Pair (A, C) is observable.

A4 Input signal $u(.)$ is unknown. The update times K_i are known and the inter-update times satisfy $(K_{i+1} - K_i - 1) \geq n+1$ for all $i \in \mathbb{N}_0$.

Note that there is no loss of generality in Assumption (**A2**), i.e. if B is not full column rank, then we can choose a matrix $\hat{B}$ whose columns form a basis for the column space of B and for each u there is a unique $\hat{u}$ such that $Bu = \hat{B}\hat{u}$. At the same time, for each $\hat{u}$ there is at least one u such that $Bu = \hat{B}\hat{u}$. Thus, the effect of the input u on the system is the same as that of $\hat{u}$. Similarly, assuming C is full row rank is only to assure that there are no redundant outputs that are obtained as a linear combination of other outputs.

B. Objectives

Under Assumptions (**A1**)-(**A4**), the objectives of this paper are the following.

1) Characterize the unobservable subspace and observability of the system (1) under unknown input.
2) Explore the implications for privacy in event-triggered networked control systems.

C. Reformulation as an impulsive system

To study the unobservable subspace of the system (1) with unknown piece-wise constant input, as described in (1c), we reformulate the system into an autonomous system with state "jumps" at the input update times. We consider the unknown piece-wise constant input as an additional state variable $\hat{u}$. Hence, the augmented state of the system is $z(k) := (x(k), \hat{u}(k))$. During the i^{th} *inter-update interval*, $\hat{u}(k)$ is constant and it "jumps" to $u(k)$ at the time steps $k \in \{K_i\}$. Thus, letting

$$\bar{A} = \begin{bmatrix} A & B \\ \mathbf{0} & I_m \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} C & \mathbf{0} \end{bmatrix},$$

we can write the dynamics (1) as

$$z(k+1) = \begin{cases} \bar{A}z(k), & \forall (k+1) \notin \mathcal{K}, \\ \begin{bmatrix} Ax(k) + B\hat{u}(k) \\ u(k+1) \end{bmatrix}, & \forall (k+1) \in \mathcal{K} \end{cases} \tag{2a}$$

$$y(k) = \bar{C}z(k). \tag{2b}$$

Thus, the full system is given by the collection of A, B, C, $\mathcal{K}$ and $u(K)$ for all $K \in \mathcal{K}$.

Note that (2) is an exact reformulation of system (1). Thus, we can study the question of observability under unknown piece-wise constant $u(.)$, with known update times, in the context of the impulsive system (2). To systematically analyze this question, we introduce the following definition.

Definition II.1. *We define the unobservable subspace of system* (2) *at time k given a horizon $w \geq k$, $\mathcal{Z}(w,k)$, as the set of all $z(k)$ for which $y(j) = \mathbf{0}$ for all $j \in [0, w]_{\mathbb{Z}}$.* •

We seek to characterize the unobservable subspace $\mathcal{Z}(w, k)$. We explore this question in the following two sections. In Section III, we address the question under the assumption of constant but unknown input. Then, in Section IV we extend the analysis to the case of piece-wise constant unknown inputs but with known update times.

III. Observability under Constant Unknown Input

The observability of the system (2) with a constant unknown input can be studied with the observability matrix associated with the pair $(\bar{A}, \bar{C})$, i.e. $O(w)$, where

$$O(w) := \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \bar{C}\bar{A}^3 \\ \vdots \\ \bar{C}\bar{A}^{w-1} \end{bmatrix} = \begin{bmatrix} C & \mathbf{0} \\ CA & CB \\ CA^2 & C(A+I)B \\ CA^3 & C(A^2+A+I)B \\ \vdots & \vdots \\ CA^{w-1} & C\sum_{i=0}^{w-2} A^i B \end{bmatrix}. \tag{3}$$

Clearly, observability of the system (1) under a constant unknown input, $\mathcal{K} = \{0\}$, is directly related to the observability of system (2) in the classical sense. In particular, it is easy to see that under a constant unknown input, $\mathcal{Z}(w, 0) = \text{Ker}(O(w))$. The following lemma characterizes the null-space of the matrix $O(w)$ and hence the unobservable subspace of the system (2) under constant unknown input.

Lemma III.1. *Suppose that Assumptions* (**A1**) *and* (**A3**) *hold. Then for all $w \geq n+1$, $z := (x, \hat{u}) \in \text{Ker}(O(w))$ iff $x = (I_n - A)^{-1}B\hat{u}$ and $\hat{u} \in \text{Ker}(C(I_n - A)^{-1}B)$. Therefore,*

$$\text{Ker}(O(w)) = \text{Im}\left(\begin{bmatrix} (I_n - A)^{-1}B \\ I_n \end{bmatrix} M_c \right),$$

where M_c is a matrix whose columns form a basis for $\text{Ker}(C(I_n - A)^{-1}B)$. Thus, under a constant unknown input, $\mathcal{K} = \{0\}$,

$$\mathcal{Z}(w, 0) = \text{Ker}(O(w)) = \text{Ker}(O(n+1)), \ \forall w \geq n+1.$$

Proof. From (3), we see that $z \in \text{Ker}(O(w))$ iff

$$CA^i x = \mathbf{0}, \ \text{for } i = 0, \tag{4a}$$

$$CA^i x + C\left(\sum_{j=0}^{i-1} A^j\right) B\hat{u} = \mathbf{0}, \ \forall i \in [1, w-1]_{\mathbb{Z}}. \tag{4b}$$

Subtracting the $(i+1)^{\text{th}}$ and i^{th} equations in (4), we get

$$CA^i B\hat{u} = CA^i (I_n - A)x, \ \forall i \in [0, w-2]_{\mathbb{Z}},$$

which can be re-arranged as

$$CA^i (B\hat{u} - (I_n - A)x) = \mathbf{0}, \ \forall i \in [0, w-2]_{\mathbb{Z}}. \tag{5}$$

Note that under Assumption (**A3**), the only vector v that satisfies $CA^i v = 0$ for all $i \in [0, n-1]_{\mathbb{Z}}$, is in fact $v =$

0. Hence (4a), (5), and Assumption (**A1**) imply that for all $w \geq n+1$, $(x, \hat{u}) \in \operatorname{Ker}(O(w))$ iff $x = (I_n - A)^{-1}B\hat{u}$ and $Cx = 0$, which is same as saying $(x, \hat{u}) \in \operatorname{Ker}(O(w))$ iff

$$x = (I_n - A)^{-1}B\hat{u} \text{ and } \hat{u} \in \operatorname{Ker}(C(I_n - A)^{-1}B). \quad (6)$$

The result now follows from the definition of M_c and the fact that $\mathcal{Z}(w, 0) = \operatorname{Ker}(O(w))$. □

Lemma III.1 provides us some insights into the structure of the unobservable subspace $\mathcal{Z}(w, 0)$. While it is natural to expect that $\mathcal{Z}(w, 0)$ would remain constant for $w \geq n+m$, the lemma in fact says that $\mathcal{Z}(w, 0)$ remains constant for all $w \geq n+1$. Further, $\mathcal{Z}(w, 0)$ for $w \geq n+1$ has specific features that we present in the following corollary.

Corollary III.2. *Suppose that Assumptions (**A1**)-(**A3**) hold and $w \geq n+1$. Then the following statements are true:*

***(a)** The dimension of $\operatorname{Ker}(O(w))$ is equal to the dimension of $\operatorname{Ker}(C(I_n - A)^{-1}B)$.*

***(b)** If $(x_1, u_1) \in \operatorname{Ker}(O(w))$ and $(x_2, u_2) \in \operatorname{Ker}(O(w))$, then either $x_1 = x_2$ and $u_1 = u_2$ or $x_1 \neq x_2$ and $u_1 \neq u_2$.*

***(c)** In particular, if $x \neq \mathbf{0}$ and $u \neq \mathbf{0}$ then $(x, \mathbf{0}) \notin \operatorname{Ker}(O(w))$ and $(\mathbf{0}, u) \notin \operatorname{Ker}(O(w))$.*

Proof. a) Assumption (**A2**) implies that the matrix

$$\begin{bmatrix} (I_n - A)^{-1}B \\ I_n \end{bmatrix}$$

has full column rank. Thus, from Lemma III.1, the dimension of $\operatorname{Ker}(O(w))$ is the same as the dimension of $\operatorname{Im}(M_c)$, which is the same as the dimension of $\operatorname{Ker}(C(I_n - A)^{-1}B)$.

b) Recall from the last step in the proof of Lemma III.1 that $(x, \hat{u}) \in \operatorname{Ker}(O(w))$ iff (6) holds. However, under Assumption (**A2**), the function $x = (I_n - A)^{-1}B\hat{u}$ from $\hat{u}$ to x is one-to-one, from which the claim follows.

c) This follows from part (b) as $(0, 0) \in \operatorname{Ker}(O(w))$. □

Remark III.3. *Notice that the above result negates the possibility of only the unknown input or only the plant state being unobservable in the initial condition $z(0)$.* •

We now go on to characterize $\mathcal{Z}(w, k)$ when $\mathcal{K} = \{0\}$. In particular, if $\mathcal{K} = \{0\}$, we can express $\mathcal{Z}(w, k)$ as

$$\begin{aligned} \mathcal{Z}(w, k) = \{ & z \in \mathbb{R}^{n+m} : \exists z_0 \in \mathbb{R}^{n+m} \text{ s.t. for (2)} \\ & z(0) = z_0,\ z(k) = z,\ y(j) = \mathbf{0},\ \forall j \in [0, w]_{\mathbb{Z}}\}, \end{aligned} \quad (7)$$

which indicates that the dimension of $\mathcal{Z}(w, k)$ is no larger than that of $\mathcal{Z}(w, 0)$. Further, we also know that $\mathcal{Z}(w, 0)$ is an $\bar{A}$-invariant subspace [23]. Thus, we can say that $\mathcal{Z}(w, k) \subseteq \mathcal{Z}(w, 0)$. But the following result says that $\mathcal{Z}(w, k) = \mathcal{Z}(w, 0)$ for all $k \geq 0$ and $w \geq n+1$.

Theorem III.4. *Consider the system (2) with a constant unknown input and suppose Assumptions (**A1**)-(**A3**) hold. Further, suppose that $z(0) \in \operatorname{Ker}(O(n+1))$. Then $z(k) = z(0)$ for all $k \geq 0$. As a consequence, $\mathcal{Z}(w, k) = \mathcal{Z}(n+1, 0)$ for all $k \geq 0$ and $w \geq n+1$.*

Proof. Lemma III.1 implies that $z(0) \in \operatorname{Ker}(O(n+1))$ iff

$$x(0) = (I_n - A)^{-1}B\hat{u}(0) \text{ and } \hat{u}(0) \in \operatorname{Ker}(C(I_n - A)^{-1}B).$$

With initial conditions $(x(0), \hat{u}(0))$, the plant state at $k = 1$ is $x(1) = Ax(0) + B\hat{u}(0)$, which is equivalent to

$$x(1) = Ax(0) + (I_n - A)x(0),$$

or $x(1) = x(0)$. Further, notice from (2) that $\hat{u}(k) = \hat{u}(1) = \hat{u}(0)$ for all $k \geq 0$. Now assume that $x(k) = x(0) = (I_n - A)^{-1}B\hat{u}(k)$, then $x(k+1) = Ax(k) + B\hat{u}(k)$ and hence $x(k+1) = x(k)$. Using mathematical induction, we conclude that $z(k) = z(0)\ \forall k \in \mathbb{N}_0$. Finally, (7) implies that $\mathcal{Z}(w, k) = \mathcal{Z}(w, 0)$ for all $k \geq 0$ and $w \geq n+1$. □

Note that $\mathcal{Z}(w, k) = \mathcal{Z}(w, 0)$ for all $k \geq 0$ and $w \geq n+1$, instead of for all $w \geq n+m$ as one might expect. Further, this result holds even if A is singular. Finally, it is interesting that if $z(0) \in \mathcal{Z}(w, 0)$ with $w \geq n+1$, then $z(k) = z(0)$ for all $k \geq 0$, which goes beyond $\bar{A}$-invariance of the set $\mathcal{Z}(w, 0)$. As we will see, this has an interesting implication for observability under an unknown piece-wise constant input, which is our next topic of discussion.

IV. Observability under Piece-wise Constant Unknown Input

We now seek to characterize $\mathcal{Z}(w, k)$ under an unknown piece-wise constant input. Analogous to (7), we can express $\mathcal{Z}(w, k)$ as

$$\begin{aligned} & \mathcal{Z}(w, k) = \\ & \{z \in \mathbb{R}^{n+m} : \exists x_0 \in \mathbb{R}^n,\ \exists u_{K_i} \in \mathbb{R}^m,\ \forall K_i \in \mathcal{K} \\ & \text{s.t. for (2)},\ x(0) = x_0,\ \hat{u}(K_i) = u_{K_i},\ \forall K_i \in \mathcal{K}, \\ & \ z(k) = z,\ y(j) = \mathbf{0},\ \forall j \in [0, w]_{\mathbb{Z}}\}, \end{aligned} \quad (8)$$

which is the set of all z such that there exist initial $x(0)$ and a piece-wise constant control with input update times $\mathcal{K}$ such that $z(k) = z$ and the output is uniformly zero on $[0, w]_{\mathbb{Z}}$. Given the extra degrees of freedom provided by u_{K_i} for $K_i \in \mathcal{K}$ it seems plausible, unlike in the constant input case, that in general $\mathcal{Z}(w, k)$ may not be a subset of $\mathcal{Z}(w, 0)$.

Remark IV.1. *Due to the causal nature of the system (2), we can say that, for all $w \in \mathbb{N}_0$ and for all $k \leq w$, $\mathcal{Z}(w, k)$ does not depend on update times greater than w. Thus, we define the truncated set of* update times up to w *as*

$$\mathcal{K}_w := \{K \in \mathcal{K} : K \leq w\} \cup \{w\} =: \{K_0, K_1, \ldots, K_{N(w)}\},$$

which is the set of all update times up to and including w. Note that we include $K_{N(w)} = w \in \mathcal{K}_w$ even if $w \notin \mathcal{K}$. However, as the input $u(w)$ can only affect the outputs $y(k)$ for $k > w$, we see that $\mathcal{Z}(w, k)$ is unaffected by whether $w \in \mathcal{K}$ or not. Hence, we can obtain $\mathcal{Z}(w, k)$ by supposing $\mathcal{K} = \mathcal{K}_w$ has only finitely many input updates. •

Now, in order to characterize $\mathcal{Z}(w, k)$, let

$$\mathbf{y}^i := (y(K_i),\ y(K_i + 1),\ \ldots,\ y(K_{i+1} - 1)),$$

which is the vector containing all the measurements in the i^{th} inter-update interval in $\mathcal{K}_w$. Then, we can write

$$\mathbf{y}^i = O(q_i)z(K_i),\ q_i := K_{i+1} - K_i,\ \forall i \in [0, N(w))_{\mathbb{Z}}. \quad (9)$$

Although $z(K_i) = (x(K_i), \hat{u}(K_i))$, note that in (9), only $x(0)$ and $\hat{u}(K_i)$ can be chosen arbitrarily. The rest of $x(K_i)$ are determined by the dynamics (2). In particular, from variation of constants, we know that

$$x(K_{i+1}) = A^{K_{i+1}-K_i}x(K_i) + \sum_{j=0}^{K_{i+1}-K_i-1}(A^jB)\hat{u}(K_i). \tag{10}$$

Now, we can say that

$$\begin{aligned}&\mathcal{Z}(w,k) = \{z \in \mathbb{R}^{n+m} : \exists v_i \in \mathbb{R}^{(n+m)},\ \forall i \in [0, N(w))_{\mathbb{Z}},\\ &\text{s.t. for (2)},\ z(K_i) = v_i,\ (10),\ z(k) = z,\\ &O(q_i)z(K_i) = \mathbf{0},\ Cx(w) = \mathbf{0},\ \forall i \in [0, N(w))_{\mathbb{Z}}\}.\end{aligned} \tag{11}$$

Now, we are ready to present our results on observability under unknown piece-wise constant control.

Theorem IV.2. *Consider system (2) with unknown piece-wise constant input and suppose Assumptions (**A1**)-(**A4**) hold. Further, let $w \in \mathbb{N}_0$ be such that $w - K_i \geq n+1$ for all $K_i \in \mathcal{K}_w \setminus \{w\}$. Finally, suppose that $k \in [0, w]_{\mathbb{Z}}$. If $z(k) = z = (x, \hat{u}) \in \mathcal{Z}(w,k)$ then $z(j) = z = (x, \hat{u})$ for all $j \in [0, w-1]_{\mathbb{Z}}$ and $x(w) = x$. As a consequence, $\mathcal{Z}(w,k) = \operatorname{Ker}(O(n+1))$ for all $k \in [0, w]_{\mathbb{Z}}$.*

Proof. Our starting point is (11). Observe that Assumption (**A4**), Lemma III.1 and the fact that $w - K_i \geq n+1$ for all $K_i \in \mathcal{K}_w \setminus \{w\}$ together imply that $O(q_i)z(K_i) = \mathbf{0}$ iff $z(K_i) \in \operatorname{Ker}(O(n+1))$ for all $i \in [0, N(w))_{\mathbb{Z}}$. Additionally, (10) implies that

$$\begin{aligned}x(K_{i+1}) &= A^{K_{i+1}-K_i}x(K_i) + \sum_{j=0}^{K_{i+1}-K_i-1} A^j(I_n - A)x(K_i)\\ &= x(K_i),\end{aligned}$$

where we have used Lemma III.1, which says $z(K_i) \in \operatorname{Ker}(O(n+1))$ implies $B\hat{u}(K_i) = (I_n - A)x(K_i)$. If $K_{i+1} - K_i \geq n+1$, then Corollary III.2 implies that $\hat{u}(K_{i+1}) = \hat{u}(K_i)$ and hence $z(K_i) = z(K_0)$ for all $i \in [0, N)_{\mathbb{Z}}$. Now, applying Theorem III.4 on each of the inter-update intervals in $\mathcal{K}_w$ in isolation and using (11), we obtain the first claim of the result. The second claim is now a consequence of (11) and the fact that $z(k) = z(K_0) \in \operatorname{Ker}(O(n+1))$. □

Note that Theorem IV.2 allows the possibility that $\mathcal{Z}(w,k)$ can be something other than $\operatorname{Ker}(O(n+1))$ for w that violate the assumption that $w - K_i \geq n+1$ for all $K_i \in \mathcal{K}_w \setminus \{w\}$. For all other w, Theorem IV.2 says that the unobservable subspace is the same. Given this, we let

$$\mathcal{Z} := \operatorname{Ker}(O(n+1)).$$

Further, for brevity, we also let $O := O(n+1)$.

Next, we want to define the *known* and the *unknown* parts of the state and the control input. To this end, letting

$$y(k : k+j) := (y(k), \ldots, y(k+j)),$$

we can write the output relation as

$$y(K_i : K_i+n) = Oz(K_i) =: O_1x(K_i) + O_2\hat{u}(K_i),\ \forall K_i \in \mathcal{K},$$

where O_1 and O_2 are the first n and last m columns of the matrix O, respectively, such that $O =: [O_1 \quad O_2]$. Assumptions (**A2**) and (**A3**) imply that O_1 and O_2 have full column rank. The claim about O_2 follows from the fact that with row operations and Cayley-Hamilton theorem, O_2 can be reduced to O_1B, which has full column rank. Hence, there exists a unique $\hat{u}(K_i)$ compatible with each pair of $x(K_i)$, and a feasible output sequence $y(K_i : K_i+n)$. Further,

$$\hat{u}(K_i) = O_2^{\dagger}\Big[y(K_i : K_i+n) - O_1x(K_i)\Big].$$

Definition IV.3. *We denote the* known *and the* unknown *parts of $z(k)$ with $r(k)$ and $\zeta(k)$, respectively, which we define as*

$$\begin{aligned}&r(0) := O^{\dagger}y(0:n),\ \zeta(0) \in \mathcal{Z},\ \textit{s.t. } z(0) = r(0) + \zeta(0)\\ &(x^{\zeta}(k), \hat{u}^{\zeta}(k)) := \zeta(k) := \bar{A}^k\zeta(0),\ \forall k \in \mathbb{N}_0\\ &(x^r(k), \hat{u}^r(k)) := r(k) := \bar{A}^{k-K_i}r(K_i),\ \forall k \in [K_i, K_{i+1})_{\mathbb{Z}},\end{aligned}$$

where

$$\begin{aligned}x^r(K_i) &:= \begin{bmatrix}I_n & \mathbf{0}\end{bmatrix}\bar{A}^{(K_i - K_{i-1})}r(K_{i-1})\\ \hat{u}^r(K_i) &:= O_2^{\dagger}\Big[y(K_i : K_i+n) - O_1x^r(K_i)\Big].\end{aligned}$$

We also call $x^{\zeta}(k)$, $\hat{u}^{\zeta}(k)$ as the unknown and $x^r(k)$, $\hat{u}^r(k)$ as the known parts in plant states and input, respectively. •

Note that $r(0)$ can only be computed after $n+1$ measurements. Similarly, for each $K_i \in \mathcal{K}$, $\hat{u}^r(K_i)$ depends on $y(K_i : K_i+n)$. This implies that $r(k)$ can only be evaluated with an initial delay of $n+1$ time-steps in each inter-update interval, that is to say that $r(k)$ can only be evaluated at time step $L_k + n$, where $L_k := \max\{K \in \mathcal{K} : K < k\}$. On the other hand, $\zeta(k)$ cannot be determined only from the measurements though we know that $\zeta(k)$ remains $\zeta(0)$ for all k. In the next result, we show that the known and the unknown parts, $r(k)$ and $\zeta(k)$, add up to $z(k)$ for all $k \geq 0$.

Corollary IV.4. *Consider the system (2) under piece-wise constant unknown input and suppose Assumptions (**A1**)-(**A4**) hold. Then $\zeta(k) = \zeta(0)$ and $z(k) = r(k) + \zeta(k)$, $\forall k \in \mathbb{N}_0$.*

Proof. Theorem III.4 ensures that $\zeta(k) = \zeta(0)$, $\forall k \geq 0$. Next, we show by induction that $z(K_i) = r(K_i) + \zeta(K_i)$, $\forall K_i \in \mathcal{K}$, which along with (2) implies that $z(k) = r(k) + \zeta(k)$ for all $k \in [K_i, K_{i+1})_{\mathbb{Z}}$, $\forall K_i \in \mathcal{K}$. By definition $r(0) \in \mathcal{Z}^{\perp}$ and hence $z(0) = r(0) + \zeta(0)$. Now suppose that $z(K_i) = r(K_i) + \zeta(K_i)$ for $K_i \in \mathcal{K}$. Then the definition of $x^r(K_{i+1})$ implies that $x(K_{i+1}) - x^r(K_{i+1}) = x^{\zeta}(K_{i+1}) = x^{\zeta}(0)$. Next, since

$$Oz(K_{i+1}) = y(K_{i+1} : K_{i+1}+n) = Or(K_{i+1}),$$

we can say that $(z(K_{i+1}) - r(K_{i+1})) \in \mathcal{Z}$. Then, Corollary III.2(b) implies that $\hat{u}(K_{i+1}) - \hat{u}^r(K_{i+1}) = \hat{u}^{\zeta}(K_{i+1}) = \hat{u}^{\zeta}(0)$ as $(x^{\zeta}(0), \hat{u}^{\zeta}(0)) \in \mathcal{Z}$. □

The known part $r(k)$ can be thought of as the estimate of the plant state and the unknown input given sufficient measurements. Theorem IV.2 and Corollary IV.4 indicate that the uncertainty about the unknown part $\zeta(k)$ cannot be reduced, from the subspace $\mathcal{Z}$, after the first $n+1$ time steps even if there are many updates to the control. However, with additional information such as the triggering rule in event-triggered control, we show in the next section that this uncertainty can be reduced. As a result, there can be a loss of privacy in event-triggered control. In contrast, in time-triggered control, there is no additional information in the update times and hence the uncertainty remains a subspace.

V. Privacy Implications for Event-Triggered Stabilization

In this section, we explore the implications of the results in Section IV for privacy in event-triggered stabilization. Through an example, we show that uncertainty about the unknown part can be reduced to a bounded subset of $\mathcal{Z}$ in finite time.

We consider a system with event-triggered state feedback transmissions from the plant to the controller over a network, in the presence of an eavesdropper (ED), as depicted in Figure 1. We assume that the eavesdropper has knowledge about the system matrices A, B and C in (1) and the event-triggering rule. While these can be known even offline, we also assume that ED has access to some online information, namely the sensor measurements and the event times $\{K_i\}$. However, we assume that ED cannot measure the plant state $x(.)$ or the control input $\hat{u}(.)$.

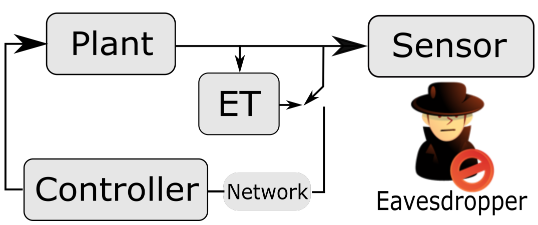

Fig. 1. Event-triggered control in the presence of an eavesdropper.

Consider system (1) with the pair (A, B) stabilizable and a matrix S such that $(A+BS)$ is schur stable. We let the input be a zero-order hold control

$$\hat{u}(k) = Sx(K_i), \ \forall k \in [K_i, K_{i+1})_{\mathbb{Z}}, \tag{12}$$

where $\mathcal{K} := \{K_i\}_{i\in\mathbb{N}_0}$ is the increasing sequence of input update times determined implicitly by an event-triggering rule. We assume that ED has knowledge of the update times K_i when they occur. However, we assume that ED has no knowledge about the matrix S or even the form of the control, except that it is piece-wise constant.

We consider the triggering rule from [24], where an event occurs at time step k, i.e. $K_{i+1} = k$, if

$$||x(K_i) - x(k)|| \geq \mu||x(k)||. \tag{13}$$

Reference [24] provides a range of values of μ for which the triggering rule (13) ensures asymptotic stabilization of the plant state to the origin. We define $\zeta = (x^\zeta, \hat{u}^\zeta) := \zeta(0)$ for brevity, then $x(k) = x^r(k) + x^\zeta$ from Corollary IV.4. As a result, the event-triggering rule can be written as: $K_{i+1} = k$ if

$$||x^r(k) + x^\zeta|| \leq \frac{1}{\mu}||x^r(K_i) - x^r(k)||. \tag{14}$$

We assume that system (1) and the update times $\mathcal{K}$ generated by the triggering rule (13) satisfy Assumptions (**A1**)-(**A4**). Assumption (**A4**) is not restrictive in this context as one could choose a smaller sampling period for time-discretizing the underlying continuous time system. With this review of event-triggering rule, we now look at the privacy implications for this stabilization task.

A. Privacy Implications

We consider plant state to be confidential information and hence a matter of privacy. Specifically, the smaller the error bound on ED's estimate of the plant state the greater is the loss of privacy. We assume that ED can accurately evaluate $x^r(k)$, the known part of the plant state $x(k)$. Hence the uncertainty in ED's estimation of the states is entirely due to the unknown part in the state. We assume that ED has access to the information $\mathcal{I}(k)$ at time k, where

$$\mathcal{I}(k) := \{A, B, C, \{y(j)\}_{j=0}^{k}, \mathcal{K}_k, \text{ET rule } \ (13)\ \}.$$

We let $\mathcal{L}(k)$ be the uncertainty set at time-step k, which is the set of all possible values of x^ζ that are compatible with information available to ED up to time k. Then, we measure the breach in privacy through the size of these uncertainty sets, denoted by $|\mathcal{L}(k)|$ at time-step k.

Notice that sensor output measurements alone cannot reduce the uncertainty set $\mathcal{L}(k)$ to something smaller than $\mathcal{X}$, which is the projection of $\mathcal{Z}$ onto the plant-state space. Thus, there is a reduction only at the event times K_i. Hence, we consider $\mathcal{L}(k)$ only for $k = K_i \in \mathcal{K}$. In particular, using (14), which is equivalent to the ET rule (13), we first define $\mathcal{S}(K_i)$ as the set of all x^ζ compatible with (14) at $k = K_i$. Thus,

$$\mathcal{S}(K_i) := \{x \in \mathcal{X} \ : \ ||x + x^r(K_i)|| \leq b(i)\},$$

where $b(i) := \frac{1}{\mu}||x^r(K_{i-1}) - x^r(K_i)||$. Then,

$$\mathcal{L}(K_i) := \cap_{j=1}^{i}\mathcal{S}(K_j). \tag{15}$$

We observe that the uncertainty sets $\mathcal{L}(K_i)$ are bounded, non-increasing with events and converge to the true value of the unknown part of the plant states as $\lim_{i\to\infty} b(i) = 0$.

Now, we give an illustrative example showing the loss of privacy about the plant state information.

B. An illustrative example

Consider system (1) with input (12) under Assumptions (**A1**)-(**A4**), with parameters

$$A = \begin{bmatrix} 1 & 0.0022 \\ -0.0044 & 1.0066 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0.0022 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$S = [1 \quad -4]$ and $\mu = 49.0636$. This value of μ ensures that the inter-event times are larger than $n+1 = 3$. In this example $\mathcal{X}$ is a line spanned by the vector $(1, 0)$. We consider the initial plant state $x(0) = (0.8, -0.4)$ and notice that $x_\zeta(0) = (1.1198, 0)$.

The evolution of the known and unknown part of plant-states in the event-triggered feedback stabilization task is shown in Figure 2. This verifies the results in Theorem IV.2 and Corollary IV.4. Also note that $x^r(k)$ approaches negative of $x^\zeta(k) = x^\zeta(0)$ asymptotically. Further, the uncertainty

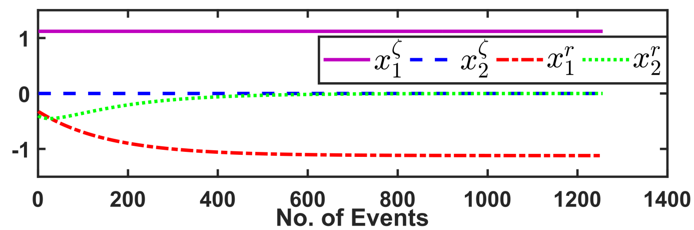

Fig. 2. Evolution of the known and the unknown parts of the plant state at the event times in the event-triggered stabilization task. Here x_i^ζ and x_i^r denote the i^{th} component in the vectors $x^\zeta(K_i)$ and $x^r(K_i)$. We see that the unknown part in plant states remains time invariant and the known part in the plant states evolves such that $\lim_{K_i \to +\infty} x^r(K_i) + x^\zeta = 0$.

sets $\mathcal{L}(K_i)$ are intervals of the line $\mathcal{X}$. In Figure 3, we show the evolution of the left and the right ends of the intervals $\mathcal{L}(K_i)$. We can see here that the length of the line

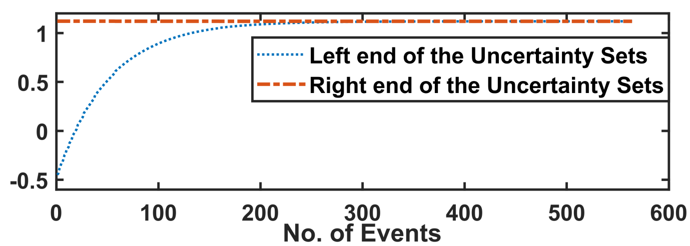

Fig. 3. The left and the right ends of the intervals of $\mathcal{X}$ that are the uncertainty sets $\mathcal{L}(K_i)$. In this figure, we see that the size of the uncertainty set, i.e. $|\mathcal{L}(K_i)|$ is bounded and decreasing with events.

segments $\mathcal{L}(K_i)$ reduces with events. Also note here that no initial estimate needs to be provided for the uncertainty set. Hence, we can see that ED can identify the unknown part of the plant state within a quantifiable bound even in finite time. Moreover, the bound shrinks with each event and converges to zero asymptotically. Thus, in this example, with the knowledge of only the system parameters A, B and C, sensor measurements, the event times and the event-triggering rule, ED is able to breach the privacy of the plant state.

VI. Conclusions

In this paper, we characterized the unobservable subspace of discrete-time LTI systems under unknown piece-wise constant inputs when the input update times are known. In particular, we showed that if the input inter-update times are long enough and if there are enough measurements in each inter-update interval, then the unobservable subspace remains fixed. We then explored the consequences of this result for privacy in event-triggered control. We showed that if an eavesdropper knows the system matrices, the input update times and the event-triggering rule then it can estimate the plant state up to a bound that decreases with time to zero.

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