One-Shot Coordination of First and Last Mode for Multi-Modal Transportation

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Abstract—**In this paper, we consider coordinated control of feeder vehicles for the first and last mode of a multi-modal transportation system. We adopt a macroscopic approach and model a geographical region as a graph with one of the nodes being an *interchange* between different modes of transportation. We model customer demands and supplies of vehicles as volumes and consider flows of vehicles. We propose one-shot problems for passenger transportation to or from the interchange within a fixed time window, under the knowledge of the demand distribution. In particular, we pose the problem of operator profit maximization through routing and allocations of the vehicles as well as pricing. With K.K.T. analysis we propose an offline method for reducing the problem size. Further, we also analyse the problem of maximizing profits by optimally locating the supply for a given total supply and present a closed form expression of the maximum profits that can be earned over all supply distributions for a given demand distribution. We also show an equivalence between optimal supply location problem and the last mode problem. Finally we present a model for determining the comparative cost of the best alternate transportation for the feeder service to be viable. We illustrate the results through simulations and also compare the proposed model with a traditional vehicle routing problem.**

Index Terms—**networked transportation systems, multi-modal transportation, first and last-mode transportation, optimization based coordination, pricing**

I. Introduction

Advances in communication, networking and computing technologies offer the opportunity to design dynamic, demand responsive and coordinated transportation systems. Such a paradigm could potentially overcome the challenging trade-off between cost of the service and the coverage of the service faced by the traditional transportation systems. This paper explores the idea of a coordinated, dynamic and demand responsive feeder service for first and last mode connectivity in a multi-modal transportation service. In particular, we consider a macroscopic *one-shot feeding* problem, in which the first and last mode services have a single hard time window and a common destination/origin respectively. This problem occurs in many scenarios such as peak-hour single destination para-transit [2], freight transportation [3], express courier systems [4], evacuation in preparation of a natural calamity [5] and management for event with a large foot-fall.

Literature Review: In the context of routing of transportation services, the vehicle routing problem (V.R.P.) [6]–[9] assumes a depot from where one or more vehicles are routed via locations where one or more types of entities are picked up and dropped of at their origin(s) and destination(s). V.R.P. has many variations like capacitated [7], multiple origin and destination [6], multiple time window [6], [8] or simultaneous pick-up and delivery [9] etc. Sharing some similarities with V.R.P. are the ride sharing problem [10]–[13] and dial-a-ride problem [14]–[16]. In these problems, the aim is to match vehicles with the passengers while maximizing operator's profits in some time windows. Recently, there is also a growing interest in routing problems with awareness of the demand and the fleet. For example, [17], [18] optimally dispatch taxis based on the location of the taxis and customer requests.

While many routing problems deal with discrete vehicles and discrete entities to be transported, macroscopic models that deal with flows of vehicles and volumes of demand and supply are also common. Though less realistic than discrete models, they allow for computationally easier solutions and greater scope for analysis and higher order planning. In the context of demand anticipative mobility, [19]–[23] aim to match demand and supply by routing autonomous vehicle flows and maximize throughput in the network through a steady-state design of the load-balancing and routing flow rates. Much of the literature on fleet routing developed in the context of uni-modal or single hop transportation services. Though multi-modal transportation has been extensively studied dynamic, demand and supply aware first and last mode service is not sufficiently studied [24].

Contributions: In this paper, we consider the first-mode or *feed-in* problem and the last-mode or *feed-out* problem, wherein all the demand has a common destination and origin, respectively. In particular, we take a macroscopic approach and pose a network flow problem. Given feeder vehicle supply and customer demand volumes at the nodes of a network, we pose the problem of maximizing the operator's profits by pricing and coordinated routing of the feeder vehicles for transporting the demand to the destination(s) in a fixed time window.

Our first contribution is a macroscopic model for the feed-in problem of maximizing the operator's profits. In the setup, after setting the prices, the problem reduces to a linear program. We obtain the optimal prices based on the notion of value of time (V.o.T.) and the idea that the perceived cost of the feeder service cannot be greater than the perceived cost for the best alternate transportation. Our second contribution is an offline (demand and supply independent) method that reduces the computational complexity of the feed-in problem by eliminating routes and the corresponding decision variables that would never be used in an optimal solution. Our third contribution is the optimal supply optimization problem for a given demand distribution and a total supply volume. This lets

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us compute the maximum profits that the operator can earn for a given total supply volume. We also provide the closed-form expression of the absolute maximum profits that the operator can earn over all supply distributions for a given demand distribution. Such a study is useful for planning and viability studies of a first and last mode feeder service. The fourth contribution of the paper is the establishment of an equivalence between the optimal supply location problem and the last-mode or feed-out problem. This equivalence enables us to utilize directly all the analytical results and methods we developed for the feed-in and supply optimization problems. The fifth contribution is a simple model for obtaining necessary conditions on the best alternate transportation for the viability of the proposed feeder service. Finally, we illustrate our results and analysis through several simulations. In the preliminary version [1] of this paper, we considered only the feed-in problem and supply location problem. Here, we additionally provide results for the feed-out problem and introduce a model for carrying out the analysis on the viability of the feeder service. Moreover, here we also provide all the proofs of the results and include additional simulation results.

Organization: The rest of the paper is organized as follows - we set up the *feed-in* problem in Section II and in Section III we describe the properties of its optimal solutions and the *off-line route-set reduction* method. In Section IV, we discuss the supply location problem and analyze maximum possible profits with a given total supply for a given demand distribution. In Section V we propose the *feed-out* problem. We present a model of best alternate transportation parameters utilised for pricing in Section VI, followed by simulation results in Section VII and conclusions. We present the proofs of all but the main theorems in appendices.

Notation: We use $\mathbb{Z}$ and $\mathbb{N}$ for the set of integers and natural numbers, respectively. We use $[a, b]_{\mathbb{Z}}$ and $(a, b]_{\mathbb{Z}}$ to denote $[a, b] \cap \mathbb{Z}$ and $(a, b] \cap \mathbb{Z}$, respectively.

II. Problem Setup

In this section, we setup the coordinated feed-in problem using a macroscopic formulation. We first describe the network or the graph variables, then the decision variables and the constraints. Finally, we summarize the full optimization problem, discuss specific challenges and aspects of it that we seek to resolve and analyze.

A. Graph Model and Routes

In this paper, we model the region for which the feeding service has to be designed as a graph $G := (V, E)$, with V and E being the set of nodes and edges in the graph, respectively. Each node $l \in V$ may represent an area or a locality. Each edge $(l, k) \in E$ represents an abstract macroscopic link from node l to node k. We assume that at each node l there is a certain demand volume, d_l, and a supply volume of feeder vehicles, S_l. All the demand is for the *interchange* node, $I \in V$, where we want all flows from all nodes in the graph G to converge. To each edge $(l, k) \in E$, we associate a per-unit flow traversal cost ρ_{lk} and an average commute time t_{lk} between the nodes. Each of these parameters is dependent on the background traffic flow, which we assume the operator knows beforehand. The service-provider needs to cater to the demand for interchange I within a fixed *destination-time* T, thus making it a *one-shot* problem.

Remark II.1. *(Supply & demand distribution and graph parameters). We assume that the demand distribution $\{d_l\}$ and graph parameters (ρ_{lk}, t_{lk}) are known and fixed. The latter assumption is justified if the change in the ambient traffic is gradual compared to the destination time T. We also assume initially that the supply distribution $\{S_l\}$ is given and can not be optimized. Finally, we let the demand at node I to be zero, though $S_I \geq 0$, in general.* •

Next, we define a route $r := (V_r, E_r)$ as a *walk* in G. For a route r, V_r is the sequence of nodes along the route, with $V_r(j)$ being the j^{th} node on the route r, whereas $E_r(j)$ is the j^{th} edge on the route r. Therefore,

$$V_r : [1, n_r]_{\mathbb{Z}} \to V, \quad (V_r(j), V_r(j+1)) \in E, \tag{1}$$

$$E_r : [1, n_r - 1]_{\mathbb{Z}} \to E, \quad E_r(j) = (V_r(j), V_r(j+1)), \tag{2}$$

where n_r is the number of nodes (possibly repeated) on the route r. We label the *origin* node of the route r as $o_r \in V$, that is $o_r = V_r(1)$. We are interested in routes with *destination* $D_r \in V$ as the interchange node, that is $D_r := V_r(n_r) = I$ and with traversal time less than T. We call such routes as *feasible routes* and define the *feasible route set* as

$$\mathbf{R} := \{r \mid t_r \leq T, \ D_r = I\}, \tag{3}$$

where $t_r := \sum_{(l,k)\in E_r} t_{lk}$ is the traversal time of the route r. Note that there exists a feasible route that passes through a node l if and only if there is a feasible route r with $o_r = l$. In general, there may be nodes through which no feasible route passes and can be removed from G without loss of generality.

For large enough T, it is possible that some routes make multiple visits to I. We call each trip to the interchange on a route as a *leg of the route*. We denote the i^{th} leg of route r by $r^i = (V_r^i, E_r^i)$ with V_r^i and E_r^i defined in the same manner as V_r and E_r respectively. We refer to the first leg of a route as its *primary leg* and all subsequent legs as its *secondary legs*. Thus each route has a primary leg but may not have a secondary leg. We identify the number of legs in a route r by $\theta_r \in \mathbb{N}$. If $\theta_r = 1$ then we say r is a *simple route*. We define a *cycle* in a leg of a route as any sub-sequence of nodes in V_r^i which starts and ends at the same node before reaching I.

Considering routes with multiple legs is particularly useful when the supply is not capable of meeting the demand in one go, in which case feeders can drop-off passengers at the interchange I and return to serve more demand to I. Let $c_r^i > 0$ denote the per-unit traversal cost on the i^{th} leg of r. Then, the per-unit traversal cost, $c_r > 0$, on a route r is

$$c_r := \sum_{(l,k)\in E_r} \rho_{lk} = \sum_{i=1}^{\theta_r} \sum_{(l,k)\in E_r^i} \rho_{lk} =: \sum_{i=1}^{\theta_r} c_r^i.$$

B. Decision Variables and Constraints

Next, we introduce the decision variables and the constraints of the problem. For each route $r \in \mathbf{R}$ we define *feeder volume*, f_r, as the volume of feeders which takes the route r. Note that the route set $\mathbf{R}$ is an exhaustive set of all feasible routes. Therefore, if a route r makes multiple visits to the Interchange I then there is a separate route in $\mathbf{R}$ for each permutation of the secondary legs of route r. Thus, we assume that the feeders f_r traverse the full route r. We call (r, i, l) as a *service tuple*, which identifies a passenger pick-up on node l on the i^{th} leg of route r. Let *allocation on a node* $f_r^i(l)$ represent the volume of demand the operator intends to pickup through service (r, i, l). Then, the *total allocation* on a node l, F_l, is

$$F_l := \sum_{r|l\in V_r} \sum_{i|l\in V_r^i} f_r^i(l). \tag{4}$$

Ideally, the total allocation at a node should be the demand that is picked-up. However, if $F_l > d_l$ then in such a case the maximum demand serviced can at-most be d_l. To identify such situations we define $\tilde{F}_l$ as the *total service* on a node l, with $\tilde{f}_r^i(l)$ as the service offered on (r, i, l). Then the service and allocations are related as follows

$$\tilde{f}_r^i(l) \leq f_r^i(l), \quad \forall\, r, i, l \tag{5a}$$

$$\tilde{F}_l := \sum_{r,i,l} \tilde{f}_r^i(l) = \min\{F_l, d_l\}, \quad \forall l \in V. \tag{5b}$$

These service constraints are economic in nature. We also have the following physical constraints on the allocations and flows

$$\sum_{l\in V_r^i} f_r^i(l) \leq f_r, \quad \forall\, i \in [1, \theta_r]_{\mathbb{Z}},\ \forall r \in \mathbf{R} \tag{6a}$$

$$\sum_{r|o_r=l} f_r \leq S_l, \quad \forall\, l \in V. \tag{6b}$$

The constraint (6a) is the *allocation constraint*, which ensures that the sum of all allocations in a leg i on a route r is at-most f_r, the feeder volume on that route, while (6b) is the *supply constraint*, which ensures that the sum of feeder volumes on all routes originating from node l, is at most the supply S_l.

Pickup times: We let $t_r^i(l)$ be the *pick-up time* for the service tuple (r, i, l). We assume that the next mode of transportation leaves at time T from the node I. Hence, the time spent on the first mode is $T - t_r^i(l)$. We assume that $t_r^i(l)$ is the last possible pick-up time for each service tuple (r, i, l). This implies that the feeders should leave their origin at the last possible time, $(T - \sum_{(l,k)\in E_r} t_{lk})$. This is justified below after we discuss pricing. Thus, we do not consider $t_r^i(l)$ as decision variables for economizing notation and to ease exposition.

Pricing: The last set of decision variables are the *prices* $p_r^i(l)$ that a unit volume of passengers pay for service on the tuple (r, i, l). We assume that the price $p_r^i(l)$ is less than the *maximum viable price*, $\bar{p}_r^i(l)$, which is the maximum price for service (r, i, l) a customer will pay.

$$p_r^i(l) \leq \bar{p}_r^i(l), \tag{7}$$

To model $\bar{p}_r^i(l)$, we utilise two concepts - *value of time* (V.o.T.), which associates a monetary cost to the travel times and *perceived cost*. In particular, we let α be the monetary value of unit time. Then, the perceived cost is $M + \alpha\tau$ for a transportation service that takes τ units of time and charges a monetary price M. For each node $l \in V$, we let $g_l := \alpha\eta_l + \zeta_l$ be the perceived cost for the *best alternate transport*, which has a travel time η_l and has a price of ζ_l. For the service (r, i, l) to be viable, the perceived cost of the feeder service should be less than or equal to perceived cost for the best alternate transportation at node l, that is

$$p_r^i(l) + \alpha(T - t_r^i(l)) \leq \zeta_l + \alpha\eta_l =: g_l.$$

Thus, the maximum viable price for the service (r, i, l) is

$$\bar{p}_r^i(l) := \zeta_l + \alpha\eta_l - \alpha(T - t_r^i(l)) = g_l - \alpha(T - t_r^i(l)). \tag{8}$$

C. Optimization Model

Next we give a model for the revenues and the cost to the operator and then we summarize the overall optimization problem from the operator's point of view. We let the revenue from service (r, i, l) be $p_r^i(l)\tilde{f}_r^i(l)$, which is the product of the price for and the volume of demand serviced by the service tuple (r, i, l). The total revenue is the sum of revenues from all the services (r, i, l). We consider two different types of costs incurred by the service-provider. First, we let the travel cost for the volume of vehicles that take the route r be $c_r f_r$, which is the product of travel cost per-unit flow and the volume of vehicles that go on route r. Second, we consider the *operational costs* (which may include incentives or commissions to the drivers and maintenance costs). We assume the operational cost is one unit for every unit of allocation on a node. Thus with f_r, $\tilde{f}_r^i(l)$, $\tilde{F}_l$, $p_r^i(l)$ and $f_r^i(l)$ as decision variables we let the *feed-in operator profit maximization problem* be

$$\begin{aligned} \max\ J &:= \sum_{(r,i,l)} p_r^i(l)\tilde{f}_r^i(l) - \left(\sum_{r\in\mathbf{R}} f_r c_r + \sum_{l\in V} F_l\right) \\ \text{s.t.}\quad & (4) - (8), \quad f_r,\ f_r^i(l) \geq 0, \\ & \forall\, r \in \mathbf{R},\ \forall\, i \in [1, \theta_r]_{\mathbb{Z}},\ \forall\, l \in V_r. \end{aligned} \tag{9}$$

Remark II.2 (Maximum viable price is the optimal price)**.** *For any fixed $f_r^i(l)$, the total profit J is a strictly increasing function of $p_r^i(l)$. If the price $p_r^i(l) \leq \bar{p}_r^i(l)$, then it has no effect on any other constraints or on other optimization variables. Therefore, the optimal price $p_r^i(l) = \bar{p}_r^i(l)$.* •

Setting $p_r^i(l) = \bar{p}_r^i(l)$, the nonlinear optimization problem (9) can be reduced to a linear program.

D. Optimal Allocations and Linear Program Formulation

From the structure of Problem (9) and as a consequence of Remark II.2, we show that the allocations $f_r^i(l)$ and passengers served $\tilde{f}_r^i(l)$ are the same, in all optimal solutions. We present the proof of this result in Appendix A1.

Lemma II.3. *(Equivalence of optimal allocations and optimal volume of passengers served). In the model (9), for any optimal solution the allocations and passengers served are the same, that is $\tilde{f}_r^i(l) = f_r^i(l)$ and hence $\tilde{F}_l = F_l \leq d_l$.* □

Given Lemma II.3, we use the terms *allocation on a node* and *service at a node* interchangeably. Similarly, we use the terms *total allocation* at a node and *total service* at a node equivalently. Further, the original nonlinear optimization problem (9) reduces to the following linear program.

$$\max_{f_r, f_r^i(l)} \bar{J} := \sum_{(r,i,l)} \beta_r^i(l) f_r^i(l) - \sum_{r \in \mathbf{R}} f_r c_r \tag{10}$$
$$\text{s.t. } (4), (6),\ F_l \le d_l,\ f_r,\ f_r^i(l) \ge 0,\ \forall (r,i,l),$$

where $\beta_r^i(l)$ is the per-unit *operator revenue* for the service (r,i,l), which we define as

$$\beta_r^i(l) := \bar{p}_r^i(l) - 1. \tag{11}$$

The -1 in the above definition is due to the assumption that operational costs are 1 unit money per-unit allocation. Thus, $\beta_r^i(l)$ is the revenue of operator from a pick-up of unit demand.

Starting with the formulation (10) we solve three problems in this paper. First we reduce the size of the linear program, and thereby computational complexity, with an offline method. Then, we define the *feed-in supply optimization problem* which extends (10) by considering the supply distribution as an optimization variable. Using this, we calculate the maximum profits for a given demand distribution. Finally, we propose the *feed-out operator profit optimization problem* and analyse its properties on lines of the above problems. Additionally, we also present a simple model for generating the perceived costs g_l for the best alternate transportation.

III. Properties of Optimal Solutions and Off-line Route Elimination

In this section, we discuss some properties of the optimal solutions of the problem (10). With these properties we reduce the size of the problem by eliminating routes in the feasible route set, $\mathbf{R}$, that would never be used in an optimal solution irrespective of the demand and supply distributions.

A. Properties of Optimal Solutions

We start by describing the cases where the constraint (6a) must be active. The following result states that the feeders on a route are allocated fully on each secondary leg. The proof is stated in Appendix A2.

Lemma III.1. *(No redundant feeders in optimal solutions). In every optimal solution, the total allocation on a secondary leg of a route r is equal to the feeder volume on that route, f_r. That is, in any optimal solution,*

$$\sum_{l \in V_r^i} f_r^i(l) = f_r, \quad \forall i \in [2, \theta_r]_{\mathbb{Z}}, \quad \forall r \in \mathbf{R}. \tag{12}$$

Next, we present necessary conditions for a route to have non-zero allocations in an optimal solution.

Proposition III.2. *(Necessary conditions for a route to be used in an optimal solution). In an optimal solution of the feed-in problem (10), if $f_r^* > 0$ for $r \in \mathbf{R}$ then the following necessarily hold.*

(a) $f_r^i(l)^* > 0$ for some $i \in [1, \theta_r]_{\mathbb{Z}}$ and $l \in V_r^i$. Further, for any (r,i,l), if $f_r^i(l)^* > 0$ then $\beta_r^i(l) \ge 0$.

(b) The route r as a whole does not make a loss, that is,

$$\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \beta_r^i(l) \ge f_r^* c_r.$$

(c) For each $i \in [2, \theta_r]_{\mathbb{Z}}$, there must exist an $l \in V_r^i$ such that $f_r^i(l)^* > 0$ and $\beta_r^i(l) \ge c_r^i$.

(d) If r is a simple route ($\theta_r = 1$) then there must exist atleast one $l \in V_r$ such that $f_r^1(l)^* > 0$ and $\beta_r^1(l) \ge c_r$. □

Proposition III.2, proven in Appendix A3, states that irrespective of the supply and demand distributions, if a route is used in an optimal solution then the following must hold for that route.

- There is a positive allocation on at least one node with each of them returning non-negative operator revenues.
- The route, as a whole, does not make a loss.
- Every secondary leg of the route should not make a loss, that is, operator revenue of a pick-up in a secondary leg is no less than the per-unit traversal cost of the leg itself.
- Every simple route used must have at least one node with non-negative per-unit operator revenue.

Using Proposition III.2, we formulate an off-line route reduction method in the next subsection.

B. Offline Route Elimination

This subsection presents the *reduced route set* for the feed-in problem (10). This set is formed by pruning out routes and the corresponding optimization variables that would have a zero allocation in every optimal solution under every possible supply and demand distributions. We obtain this by application of the individual properties in Lemma III.1 and Proposition III.2, after eliminating the dependence on f_r and $f_r^i(l)$. We first define the reduced route set $\bar{\mathbf{R}}$, then show that any route with $f_r^* > 0$ in every optimal solution to the feed-in problem (10) belongs to $\bar{\mathbf{R}}$.

$$w_r^i := \max_{l \in V_r^i} \{\max\{\beta_r^i(l), 0\}\} - c_r^i, \quad \bar{\mathbf{R}} := \mathbf{R}_1 \cup \mathbf{R}_2 \tag{13}$$

$$\mathbf{R}_1 := \{r \in \mathbf{R} \mid \theta_r = 1, w_r^1 \ge 0\} \tag{14}$$

$$\mathbf{R}_2 := \{r \in \mathbf{R} \mid \theta_r > 1, \sum_{i=1}^{\theta_r} w_r^i \ge 0,\ w_r^i \ge 0, \forall i > 1\} \tag{15}$$

Theorem III.3. *(Optimal solutions to the feed-in problem use only the routes from the reduced route set). For the optimization problem (10), every optimal solution for every demand and supply distribution is guaranteed to have $f_r^* = 0$ and consequently $f_r^i(l)^* = 0$ over all legs i of r, $\forall\ r \notin \bar{\mathbf{R}}$.*

Proof. We prove this result by contradiction - hence let there exist an optimal solution such that $f_r^* > 0$ for some $r \notin \bar{\mathbf{R}}$. If $\theta_r = 1$, then it satisfies Proposition III.2(d) and as a consequence $\exists l \in V_r^1$ s.t. $\beta_r^1(l) \ge c_r = c_r^1$, which implies $w_r^1 \ge 0$. Therefore $r \in \mathbf{R}_1$. Now, if $\theta_r > 1$, then r satisfies Proposition III.2(c). Hence

$$w_r^i = \max_{l \in V_r^i} \{\max\{\beta_r^i(l), 0\}\} - c_r^i \ge 0, \quad \forall i > 1.$$

Further, r must also satisfy Propositions III.2(a) and III.2(b), that is,

$$f_r^* c_r \le \sum_{i=1}^{\theta_r} f_r^i(l)^* \beta_r^i(l) \le \sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} \max\{\beta_r^i(l), 0\} f_r^i(l)^*$$

$$\leq \sum_{i=1}^{\theta_r} \max_{l\in V_r^i}\{\max\{\beta_r^i(l)\},0\} f_r^*,$$

where we have used the fact that $f_r^i(l)^* > 0$ only if $\beta_r^i(l) \geq 0$ for the second inequality and the third inequality follows from (6a). Hence, r must satisfy $\sum_{i=1}^{\theta_r} w_r^i \geq 0$ which implies $r \in \mathbf{R}_2$. Therefore, in either case, $r \in \mathbf{R}_1 \cup \mathbf{R}_2 = \bar{\mathbf{R}}$. This is a contradiction. □

In the proof we don't explicitly check Proposition III.2(a) for the route but one can verify that $\forall\ r \in \bar{\mathbf{R}}$ there exists $\beta_r^i(l) \geq 0$ where $f_r^i(l)^* > 0$ is possible for some supply and demand distribution. Thus, replacing $\mathbf{R}$ with $\bar{\mathbf{R}}$ in Problem (10) causes no *approximation* or *loss* of any optimal solutions.

Remark III.4. *(Reduced route set reaches a constant as the destination time T is increased). There exists a time T^* such that for all $T \geq T^*$, the set $\bar{\mathbf{R}}$ is the same. This is because even though the set $\mathbf{R}$ includes more and more routes as T increases, the time from pickup at a node to drop off at interchange I cannot exceed a certain value to maintain $\beta_r^i(l) \geq 0$. Furthermore, even for pick-ups with higher pickup times, higher costs would render them unprofitable. Thus $\bar{\mathbf{R}}$ do not have such routes. This is particularly useful as the set $\mathbf{R}$ and problem* (10) *keeps growing with T, whereas the size of* (10) *with $\bar{\mathbf{R}}$ instead or $\mathbf{R}$ does not grow for $T \geq T^*$.* •

IV. Feed-in Supply Optimization problem

Our main focus till the previous section had been to maximize operator's profits for the feed-in problem, given a supply and demand distribution. However, an operator may also be interested in "aligning" the supply to the demand distribution so that profits increase. Thus, we next focus on the problem of optimizing the supply distribution to maximize profits given the demand distribution and the total available supply s, where we assume that $\{S_l\}$ are also optimization variables. Then, we let the *feed-in supply optimization problem* be

$$\max_{S_l, f_r, f_r^i(l)} \bar{J} := \sum_{(r,i,l)} \beta_r^i(l) f_r^i(l) - \sum_{r\in\mathbf{R}} f_r c_r$$
$$\text{s.t. } (4), (6),\ F_l \leq d_l,\ \sum_{l\in V} S_l \leq s,\ f_r,\ f_r^i(l),\ S_l \geq 0$$
$$\forall\ r \in \mathbf{R},\ \ \forall\ i \in [1,\theta_r]_{\mathbb{Z}},\ \forall\ l \in V_r. \quad (16)$$

For analysing the maximum profit as a function of the total supply s, we make the following assumptions.

(A1) The demand distribution $\{d_l\}_{l\in V}$ is fixed. Also, $d_l > 0$ for each node $l \neq I$ and $d_I = 0$.

(A2) For each node l in the graph $\exists\ r \in \mathbf{R}$ s.t. $o_r = l$, $\theta_r = 1$ and $\beta_r^1(l) - c_r^1 \geq 0$.

1) General Properties for Feed-in Supply Optimization: In the following proposition, we analyse the properties of the optimal solutions of *feed-in supply optimization problem* (16) and show that there is no loss of generality in the Assumptions **(A1)** and **(A2)**. We present the proof in Appendix B1.

Proposition IV.1. *(Properties of optimal supply distributions and allocations). In every optimal solution to the problem* (16)*, the following hold:*

(a) If $f_r^* > 0$ then $r \in \bar{\mathbf{R}}$

(b) For each $r \in \bar{\mathbf{R}}$, $f_r^1(o_r)^* = f_r^*$. Consequently,

$$\sum_{l\in V_r^i} f_r^i(l)^* = f_r^*, \quad \forall i \in [1,\theta_r]_{\mathbb{Z}}, \quad \forall r \in \bar{\mathbf{R}}, \quad (17)$$

and no route originating from I is used.

(c) For a route $r \in \bar{\mathbf{R}}$ with $f_r^* > 0$, $\beta_r^1(o_r) \geq c_r^1$. Consequently, $\forall i \in [1,\theta_r]_{\mathbb{Z}}$, $\exists\ l \in V_r^i$ such that $f_r^i(l)^* > 0$. Moreover, for (r,i,l), $f_r^i(l)^* > 0$ only if $\beta_r^i(l) \geq c_r^i$.

(d) If a node l does not satisfy the property in **(A2)** *then $f_r^i(l)^* = 0$ for all (r,i,l) that serve the node l.*

(e) A route r with a cycle in the first leg is not used, that is $f_r^* = 0$.

(f) If $s \leq \sum_{l\in V} d_l$ and **(A2)** *holds then $\sum_{r|o_r=l} f_r^* = S_l \leq d_l,\ \forall l \in V$.* □

From Proposition IV.1 we see that there is no loss of generality in the Assumptions **(A1)** and **(A2)**. This is because if for a node l, $d_l = 0$ then, by Proposition IV.1(b), all routes r originating at l have zero flow ($f_r = 0$) in every optimal solution. Similarly, Proposition IV.1(d) says that in every optimal solution there is no allocation on nodes that violate Assumptions **(A2)**.

With Proposition IV.1(d) one can eliminate the nodes that do not follow assumption **(A2)**. Also, as a consequence of Propositions IV.1(b) and IV.1(c), one can eliminate the route flow variables f_r and remove routes without a *profitable* pickup at their origins. With Proposition IV.1(e) we can eliminate every route with a cycle in the first leg. Thus we construct the *reduced route set* for (16), $\mathbf{R}^-$, as

$$\mathbf{R}^- := \{r \in \bar{\mathbf{R}} | o_r \neq I, \beta_r^1(o_r) \geq c_r^1,\ \text{no cycles in } r^1\}. \quad (18)$$

Further, using Proposition (IV.1) and (17), we can solve (16) with strict equality in the constraints of (6a) and (6b). Thus, we can reduce (16) to an optimization problem over decision variables $f_r^i(l)$, the allocations, and S_l, the supply at a node. This elimination of the variables f_r leads to a significant reduction in the number of optimization variables, specifically equal to the number of routes in $\mathbf{R}^-$. As a result, we can express the supply optimization problem as

$$\max_{S_l,\ f_r^i(l)} \bar{J} = \sum_{(r,i,l)|r\in\mathbf{R}^-} (\beta_r^i(l) - c_r^i) f_r^i(l) \quad (19)$$
$$\text{s.t. } (4),\ F_l \leq d_l,\ \sum_{l\in V_r^i} f_r^i(l) = f_r,\ \sum_{r|o_r=l} f_r = S_l,\ \sum_{l\in V} S_l \leq s,$$
$$f_r^i(l),\ S_l \geq 0,\ \forall\ r \in \mathbf{R}^-,\ \ \forall\ i \in [1,\theta_r]_{\mathbb{Z}},\ \forall\ l \in V_r^i.$$

This is a simpler problem to solve for a sequence of values of s than (16). Also, as we show in Section V, this formulation makes the feed-out problem computationally simpler.

2) Absolute Maximum Profits: With the objective (19), we can also analyse the absolute maximum profits an operator can earn, over all supply distributions, for a given demand distribution. Quantification of the absolute maximum profits is useful for determining feasibility or profitability of the service from the operator's perspective. To arrive at the value of *absolute maximum profits*, denoted by $J_{\max}$ from here on, we assume that supply is sufficient, i.e. $s \geq \sum_l d_l$. We also denote the set of simple routes originating at l as

$$\sigma(l) := \{r \in \mathbf{R} \mid o_r = l,\ \theta_r = 1\} \quad (20)$$

and we denote the set of simple routes with the maximum rate of profits for a pickup at l by

$$\mathcal{R}(l) := \underset{r\in\sigma(l)\cap\mathbf{R}^-}{\operatorname{argmax}} \{\beta_r^1(l) - c_r\}. \tag{21}$$

Lemma IV.2. *For each node $l \in V$, $\beta_r^1(l) - c_r > \beta_{\bar{r}}^i(l) - c_{\bar{r}}^i$ for all $r \in \mathcal{R}(l)$, $\bar{r} \notin \mathcal{R}(l)$ and $i \in [1, \theta_{\bar{r}}]_{\mathbb{Z}}$. Further, $\forall r \in \mathcal{R}(l)$ the perceived cost $(\alpha t_r + c_r)$ is the least from node l.* □

The lemma is proved in Appendix B2. Next, we state the properties of optimal solutions of (19) for sufficient supply.

Theorem IV.3. *(Properties of optimizers under sufficient supply). If $s \geq \sum_{l\in V} d_l$, then all optimal solutions of (19) satisfy*

(a) If $r \notin \mathcal{R}(o_r)$ then $f_r^* = 0$. Further, for each $l \in V \setminus \{I\}$, $F_l^* = \sum_{r\in\mathcal{R}(o_r)} f_r^* = d_l$ and $S_l^* \geq d_l$.

(b) The maximum profits over all supply distributions is

$$J_{max} = \sum_{l\in V} d_l \max_{r\in\mathcal{R}(l)} \{\beta_r^1(l) - c_r^1\}. \tag{22}$$

Proof. **(a):** As $s \geq \sum_l d_l$, consider a solution where, $\sum_{r\in\mathcal{R}(l)} f_r^1(l) = d_l$, $f_r^i(l) = 0$ for all $r \notin \mathcal{R}(l)$ for each $l \in V$, and $S_l = \sum_{r\in\mathcal{R}(l)} f_r^1(l)$, $\forall\, l \neq I$, and $S_I = s - \sum_l d_l$. One can verify that such a solution is feasible under Assumption **(A2)**. From Lemma IV.2, we know that $\beta_r^1(l) - c_r > \beta_q^i(l) - c_q^i$ for all $r \in \mathcal{R}(l)$ and $q \notin \mathcal{R}(l)$. Then the structure of the objective function (19) implies that this solution is also optimal.

(b): Given the part (a), we now see that the maximum profits must satisfy (22), in which the term indexed by l corresponds to the profits from node l. □

This theorem gives the absolute maximum profits for a given demand distribution over all supply distributions. The value of $J_{\max}$ is easily computable with knowledge of maximum rates of profit for simple routes and demand distribution.

V. One shot feed-out

In this section, we propose the *one-shot feed-out* problem on the lines of the *one-shot feed-in* problem. The goal is to drop-off passengers at different destinations from a single origin within a single, fixed time window, $\hat{T}$. We assume a fixed demand distribution $\{\hat{d}_l\}$, where $\hat{d}_l$ represents the demand from the *interchange* node I to node $l \in V$. We also assume that the total available supply is s and concentrated at I with $S_l = 0, \forall l \neq I$. Hence, we are interested in routes with $o_r = I$. We define the set of feasible routes as

$$\hat{\mathbf{R}} := \{r | (1) - (2),\ o_r = I,\ t_r \leq \hat{T}\}.$$

Let $\hat{f}_r$ denote the *feed-out flow* for a route $r \in \hat{\mathbf{R}}$, let $\hat{f}_r^i(l)$ represent *feed-out allocation* for a service (r, i, l) and let the *total feed-out node allocation* for node l be denoted by $\hat{F}_l$. Then, the constraints on these variables are

$$\sum_{r\in\hat{\mathbf{R}}} \hat{f}_r \leq s \tag{23a}$$

$$\hat{F}_l := \sum_{r,i} \hat{f}_r^i(l) \leq \hat{d}_l, \quad \forall l \in V \tag{23b}$$

$$\sum_{l\in V_r^i} \hat{f}_r^i(l) \leq f_r, \quad \forall i \in [1, \theta_r]_{\mathbb{Z}},\ \forall r \in \hat{\mathbf{R}}. \tag{23c}$$

These constraints are exactly analogous to the ones in the feed-in problem. As in the feed-in problem, one could again demonstrate that the demand serviced for any node is the same as the total feed-out node allocation (see Lemma II.3). Thus, we ignore the service variables.

We let $\hat{p}_r^i(l)$ be the *feed-out price* a unit of passengers pays for the service (r, i, l). We assume an operational cost of 1 unit money per unit allocation. We can then define the *drop-off operator revenue* on the lines of (11) as

$$\hat{\beta}_r^i(l) := \hat{p}_r^i(l) - 1. \tag{24}$$

Again, as in the feed-in problem, we can set the prices independent of the flows and allocations. Thus with the optimization variables $\hat{f}_r$, $\hat{f}_r^i(l)$ we can write the *feed-out operator profit maximization problem* as

$$\max_{\hat{f}_r, \hat{f}_r^i(l)} \hat{J} = \sum_{r,i,l} \hat{\beta}_r^i(l) \hat{f}_r^i(l) - \sum_r \hat{f}_r c_r$$
$$\text{Subject to: } (23),\ \hat{f}_r, \hat{f}_r^i(l) \geq 0, \quad \forall (r, i, l). \tag{25}$$

Now we fix the price $\hat{p}_r^i(l)$. We denote the drop-off time for the service tuple (r, i, l) by $\hat{t}_r^i(l)$. Let $\hat{\eta}_l$ and $\hat{\zeta}_l$ are the best transportation time and costs from I to l respectively. Then the optimal price for a (r, i, l) using perceived costs is

$$\hat{p}_r^i(l)^* = \alpha(\hat{\eta}_l - \hat{t}_r^i(l)) + \hat{\zeta}_l, \quad \forall (r, i, l). \tag{26}$$

Further, analogous to the feed-in problem, the drop-off time $\hat{t}_r^i(l)$ should be the least possible as the price is reduced otherwise. Hence, we assume that the service starts at $t = 0$. This implies that $\hat{t}_r^i(l)$ is the traversal time from I to l along the route r with the drop-off on the service (r, i, l).

A. Equivalence to feed-in supply optimization problem

In this subsection, we show that for the feed-out problem (25), an equivalent supply optimization feed-in problem exists with the same optimization value and related optimizers. We first propose a set of supply distributions that always contain an optimizer of the problem (16).

Lemma V.1. *For total supply s, the set of supply distributions*

$$\mathcal{S}(s) := \{\{S_l\} | S_I = \max\{0, s - \sum_{l\neq I} d_l\}\} \tag{27}$$

always contains an optimizer for the feed-in supply optimization *problem defined in* (16). □

The proof of this lemma follows from Proposition IV.1(f) and Theorem IV.3(a). Next, for the feed-out problem (25) on the graph $\hat{G} = (\hat{V}, \hat{E})$, we construct an equivalent feed-in supply optimization problem with the following construction. We express the construction through the following assumptions.

- **(A3)** Let the graph $G = (V, E)$ be such that $V = \hat{V}$, edge $(l, k) \in \hat{E}$ iff $(k, l) \in E$ and $(\hat{\rho}_{lk}, \hat{t}_{lk}) = (\rho_{kl}, t_{kl})$ $\forall (l, k) \in \hat{E}$.
- **(A4)** The supply distribution for (16) is chosen from the supply set (27). Also, $\{\hat{d}_l\} = \{d_l\}$

(A5) Let $T = \hat{T}$. Also, the best alternate travel times and costs are the same, i.e. $\eta_l = \hat{\eta}_l$ and $\zeta_l = \hat{\zeta}_l$.

Now we will give a mapping for the route set $\mathbf{R}$ and $\hat{\mathbf{R}}$.

Remark V.2. *(Equivalent graphs). Given a route r on graph $\hat{G}$, let $\phi_{\hat{G}G}(r) := \bar{r}$, a route in G (defined by **(A3)**) where $\theta_r = \theta_{\bar{r}}$, $V_r(i) = V_{\bar{r}}(n_{\bar{r}} - i + 1)$ and $E_{\bar{r}}(n_{\bar{r}} - i) = (V_r(i+1), V_r(i))$. Then, $\forall\ r \in \hat{\mathbf{R}}$, $\exists \phi_{\hat{G}G}(r) = \bar{r} \in \mathbf{R}$ with $c_r = c_{\bar{r}}$ and every service tuple (r, i, l) of r mapping to $(\bar{r}, \theta_r - i + 1, l)$.* •

With this remark we can now show that $\hat{\beta}^i_r(l)$ and $\beta^{\theta_r - i+1}_{\bar{r}}(l)$ are the same under the assumptions given above.

Lemma V.3. *For the feed-out problem and the corresponding feed-in problem, $\hat{\beta}^i_r(l) = \beta^{\theta_r - i+1}_{\bar{r}}(l)$, where $\bar{r} = \phi_{\hat{G}G}(r)$.* □

The proof of this Lemma is given in Appendix C1. Next we show equivalence of the two problems.

Theorem V.4. *(Equivalence of the feed-out problem and the supply location feed-in problem). Under Assumptions **(A3)**-**(A5)**, the feed-out problem defined in (25) can be represented by an equivalent supply optimization feed-in problem (16) with $\hat{J}^*(s) = \bar{J}^*(s)$, $\forall s \geq 0$. Further, for all optimal solutions*

(a) $\hat{f}^i_r(l)^* = f^{\theta_r - i+1}_{\bar{r}}(l)^*$, $\forall (r, i, l)$ and $\bar{r} = \phi_{\hat{G}G}(r)$,

(b) $\hat{f}^*_r = f^*_{\bar{r}}$, $\bar{r} = \phi_{\hat{G}G}(r)$, $\hat{F}^*_l = F^*_l$, $\forall\ l \in V$.

Proof. Each pair of $r \in \hat{\mathbf{R}}$ and $\phi_{\hat{G}G}(r) = \bar{r} \in \mathbf{R}$ satisfy the relationship given in Remark V.2. Using Lemma V.3, we see that $\hat{\beta}^i_r(l) = \beta^{\theta_r - i+1}_{\bar{r}}(l)$ and given $T = \hat{T}$ we conclude that the cost functions are equivalent, i.e. $\bar{J} \equiv \hat{J}$ under the assumption $\hat{f}^i_r(l) = f^{\theta_r - i+1}_{\bar{r}}(l)$. Hence, it is sufficient to prove equivalence of the constraints in both problems.

Leg Constraints: In both problems given flows $\hat{f}_r$ and $f_{\bar{r}}$ the constraint for leg i and route r in *feed-out* problem (23c) and the constraint for leg $\theta_r - i + 1$ and route $\bar{r}$ for the feed-in supply optimization (see (19)) are equivalent.

Demand Constraints: Constraint (23b) and Constraint $F_l \leq d_l$ are equivalent under the assumption that $d_l = \hat{d}_l$.

Supply Constraints: To show this equivalence, we let $\{\hat{S}_l\}_l$ be the supply distribution at the end of *feed-out*. We know that for any route r, the flow terminates at D_r. Therefore, the final supply located at any node l is $\hat{S}_l = \sum_{r|D_r = l} \hat{f}_r$. Therefore, Constraint (23a) can be rewritten as

$$\sum_r \hat{f}_r = \sum_l \sum_{r|D_r=l} \hat{f}_r = \sum_l \hat{S}_l \leq s,$$

which implies, $\hat{S}_l = \sum_{r|D_r=l} \hat{f}_r$, and $\sum_l \hat{S}_l \leq s$. Now, with the assumption that supply distribution for *feed-in* is chosen from the set (27), one can see that constraint (6b) along with $\sum_{l \in V} S_l \leq s$ are equivalent to ones stated above as consequences of Proposition IV.1(f) for $s \leq \sum_l d_l$ and Theorem IV.3(a) with Lemma V.1 for $s \geq \sum_l d_l$. □

Theorem V.4 establishes the equivalence of the feed-out problem on the graph $\hat{G}$ to the supply optimization problem on the graph G, formed using **(A3)**. Thus, one may solve either problem and obtain a solution to the feed-out problem. More importantly, many of the properties and results of supply optimization problem apply for the feed-out problem.

B. Route Pruning and Absolute Maximum Profits

Here, we state some properties of optimal solutions of feed-out problem. One can prove them using the results for the feed-in and supply optimization feed-in problems and by using the equivalence stated in Theorem V.4.

Corollary V.5. *(Necessary conditions for a route to be used in an optimal solution). In every optimal solution to the problem (25), if $\hat{f}^*_r > 0$, $r \in \hat{\mathbf{R}}$, then*

(a) For all legs of the route $r \in \hat{\mathbf{R}}$ we have

$$\hat{f}^*_r = \sum_{l \in V^i_r} \hat{f}^i_r(l)^*, \quad \forall\ i \in [1, \theta_r]_{\mathbb{Z}}.$$

(b) $\hat{f}^i_r(l)^* > 0$ for some $i \in [1, \theta_r]_{\mathbb{Z}}$ and $l \in V^i_r$. Also, if $\hat{f}^i_r(l)^* > 0$ for any (r, i, l) then $\hat{\beta}^i_r(l) \geq 0$.

(c) The route as well as its every leg is profitable i.e.

$$\sum_{i,l} \hat{\beta}^i_r(l) \hat{f}^i_r(l)^* \geq \hat{f}^*_r c_r.$$

Also, $\forall\ i \in [1, \theta_r]_{\mathbb{Z}}$, $\exists\ l \in V^i_r$ where $\hat{f}^i_r(l)^* > 0$ with $\hat{\beta}^i_r(l) \geq c^i_r > 0$.

(d) The destination of the route is profitable i.e. $\hat{\beta}^{\theta_r}_r(D_r) \geq c^{\theta_r}_r$. Also, $\hat{f}^{\theta_r}_r(D_r)^* = \hat{f}^*_r$.

(e) The route doesn't contain a cycle in the final leg and doesn't terminate in I . □

Corollary V.5 states the necessary conditions for a route to be utilised in some optimal solution. As was stated for Propositions III.2 and IV.1, this corollary also presents properties independent of demand distribution or total supply. Thus, combining all properties that are satisfied for routes used in any optimal solutions, reduced route set $\hat{\mathbf{R}}^-$ is obtained as

$$\hat{\mathbf{R}}^- := \{r \in \hat{\mathbf{R}} | \bar{r} \in \mathbf{R}^-, \bar{r} = \phi_{\hat{G}G}(r)\} \tag{28}$$

The route set $\hat{\mathbf{R}}^-$ contains all routes used in any optimal solution to the feed-out problem (25). Also utilising Corollary V.5(a) the equivalent reduced problem for (25) is

$$\begin{aligned} &\max_{\hat{f}^i_r(l)} \ \hat{J} = \sum_{r,i,l} (\hat{\beta}^i_r(l) - c^i_r) \hat{f}^i_r(l) \\ &\text{Subject to: } \sum_{l \in V^i_r} \hat{f}^i_r(l) = \hat{f}^{\theta_r}_r(D_r), \sum_{r,i} \hat{f}^i_r(l) \leq \hat{d}_l, \\ &\sum_{r \in \hat{\mathbf{R}}^-} \hat{f}^{\theta_r}_r(D_r) \leq s, \ \hat{f}^i_r(l) \geq 0, \ \forall (r, i, l). \end{aligned} \tag{29}$$

We next define the set of simple routes for a destination $l \in V$ that have the highest per-unit allocation profits.

$$\hat{\mathcal{R}}(l) := \underset{r \in \hat{\mathbf{R}}^- | D_r = l, \ \theta_r = 1}{\operatorname{argmax}} \{\hat{\beta}^i_r(l) - c_r\}. \tag{30}$$

Using this definition, we state some properties of solutions to (29) that depend on the supply s and also give the maximum profits an operator can earn for the feed-out problem.

Corollary V.6. *(Dependence of optimal solutions on total supply). For an optimal solution of (29),*

(a) If $s \leq \sum_l \hat{d}_l$, then $\sum_r \hat{f}^{\theta_r}_r(D_r)^* = s$.

(b) If $s \geq \sum_l \hat{d}_l$ *and* $\hat{f}_r^{\theta_r}(D_r)^* > 0$ *then* $r \in \hat{\mathcal{R}}(l)$ *with* $\sum_{r \in \hat{\mathcal{R}}(l)} \hat{f}_r^{\theta_r}(D_r)^* = \hat{d}_l$.

(c) The absolute maximum profits over all s *are*

$$\hat{J}_{\max} = \sum_l \hat{d}_l \max_{r \in \hat{\mathbf{R}}^- | D_r = l,\ \theta_r = 1} \{\hat{\beta}_r^i(l) - c_r\}. \quad (31)$$

As one can see, $\hat{\mathcal{R}}(l)$ has properties similar to those proposed in Lemma IV.2 Also, the absolute maximum profits are similar in nature to that in supply location feed-in problem.

VI. Best Alternate Transport and its Effect onthe Reduced Route Set

In this section, we present a simple model of the perceived costs for the best alternate transportation $\{g_l\}$ and explore the their effect on the feasibility of the feeder service.

1) Modelling the Perceived Costs of the Best Alternate Transportation: In order to systematically generate g_l for each node $l \in V$, we first assume that the best alternate transport available in the region costs bc_r and time t_r along a route $r \in \mathbf{R}$. The *cost-factor* $b \geq 0$ signifies the cost to a passenger of the best alternate transportation relative to the feeder service. For simplicity, we assume it to be the same through out the service area. Then, we let

$$r(l,b)^* \in \underset{r \in \sigma(l)}{\operatorname{argmin}}\{\alpha t_r + bc_r\} \quad (32)$$

$$g_l(b) := \alpha \eta_l + \zeta_l,\ \eta_l = t_{r(l,b)^*},\ \zeta_l = bc_{r(l,b)^*}, \quad (33)$$

where $r(l,b)^*$ is a route that the best alternate transport uses from node l to node I, while η_l, ζ_l and $g(l(b)$ are the travel time, cost and the perceived cost of the best alternate transport from node l to node I.

Remark VI.1. *(Effect of the cost-factor on best alternate transportation). For a fixed* T*, as there are finitely many routes, the perceived cost* $g_l(b)$ *is a piecewise-linear, increasing, concave and unique function of* b *for each node* $l \in V$*. Further, at* b*, the slope of* $g_l(b)$ *is equal to* $c_{r(l,b)^*}$ *and the* g *intercept is* $t_{r(l,b)^*}$*. Thus,* η_l *and* ζ_l *are also unique for each* b *except where the slope of* $g_l(b)$ *changes. For* $b = 0$ *and* $b = \infty$*, the routes* $r(l,b)^*$ *are the fastest and cheapest, respectively. For any* b*,* $r(l,b)^*$ *is such that* $t_{r(l,b)^*} \geq t_{r(l,0)^*}$ *and* $c_{r(l,b)^*} \geq c_{r(l,\infty)^*}$*.* •

2) Viability of Feeder Service: We first present a necessary condition on the value of b for the reduced route set $\bar{\mathbf{R}}$ to be non-empty and as a result for the feeder service to be viable.

Lemma VI.2. *(Necessary condition on* b *for viability of feeder service). If* $g_l(b)$ *is given by* (33)*, for each* $l \in V$*, then the reduced route set* $\bar{\mathbf{R}}$ *is non-empty only if* $b > 1$*.* □

We present the proof in Appendix D1. Next, we check for the existence of multi-legged routes in $\bar{\mathbf{R}}$ given b. We denote $c^*(I,l)$ as the cheapest cost to go from I to l in the graph for the following proposition. Its proof is given in Appendix D2.

Proposition VI.3. *(Necessary value of* b *for* $\bar{\mathbf{R}}$ *to contain multi-legged routes). Consider the following statements*

(a) $\exists\, r \in \bar{\mathbf{R}}$ *such that* $\theta_r > 1$

(b) $\exists\, r \in \bar{\mathbf{R}}$ *with* $o_r = I$ *and* $\theta_r = 1$

(c) $g_l(b) \geq g_l(1) + c^*(I,l) + 1$*, for some* $l \in V$*,* $l \neq I$

(d) $b \geq \left(1 + \dfrac{1 + c^*(I,l) + \alpha(t_{r(l,1)^*} - t_{r(l,\infty)^*})}{c_{r(l,1)^*}}\right) =: b_l^*$ *for some* $l \in V$*,* $l \neq I$*.*

Then, (a) $\Rightarrow$ *(b)* $\Rightarrow$ *(c)* $\Rightarrow$ *(d). Also, (c)* $\Rightarrow$ *(b) for sufficiently large* T*.* □

With this proposition, given the graph G, the value of time α and the value of b, the operator can evaluate viability of the feeder service. Also, Proposition VI.3(c) and VI.3(d) for specific nodes l may be interpreted as necessary conditions for a traditional V.R.P. service to that node to be viable. Note that we give two necessary conditions in Proposition VI.3, namely parts (c) and (d), because the condition VI.3(c) requires the computation of $g_l(b)$ for each b. In comparison, condition VI.3(d) is a computationally simpler relation but provides a bound lower than the one in condition VI.3(c).

VII. Results

Since both feed-in problem (10) and feed-out problem (29) are linear programs, we utilized CVXpy [25] for simulations.

1) Simulation Setup: We used the 24 node graph in Figure 1 for the simulations of the feed-in problems. The *Interchange* node, $I = 23$ and *destination-time* is $T = 30$. The feasible route set $\mathbf{R}$ has 37283 routes with 274411 variables. We also note that all nodes satisfy Assumption **(A2)**. We assume V.o.T. to be $\alpha = 0.5$.

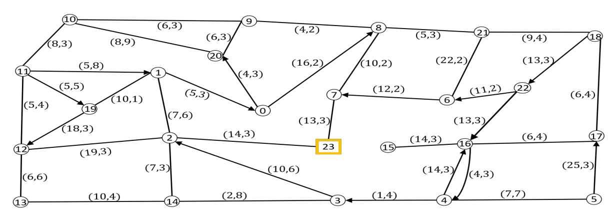


Fig. 1: Graph for simulation of feed-in problems. Numbers in the circles represent node index, an arrow between nodes l and k indicates a directed edge from l to k and a line without an arrow between nodes l and k indicates a bi-directional edge. The tuple (a,b) on the edge (l,k) represents (ρ_{lk}, t_{lk}) and the Interchange node $I = 23$ is marked in square.

Given the cost-factor b, we utilised (33) to generate the best alternate transportation time and cost, η_l, ζ_l respectively, for each node l and (8) to generate prices for each service tuple (r,i,l). Figure 2 shows the number of routes in $\bar{\mathbf{R}}$ as a function of b (with a step size of 0.01) for the graph in Figure 1. We note that the first route with origin as I

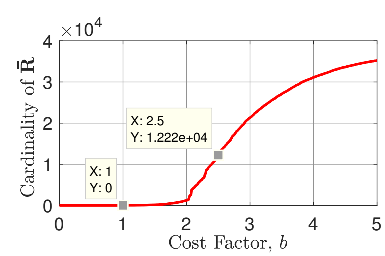
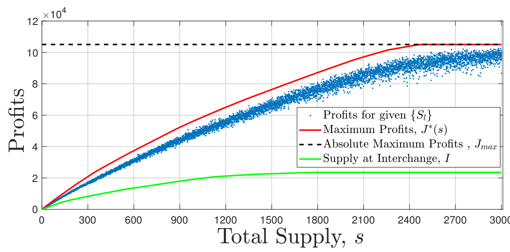
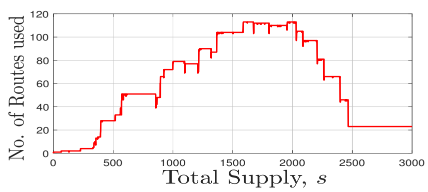


Fig. 2: Variation of Number of Routes in $\bar{\mathbf{R}}$ versus b for $\alpha = 0.5$ and $T = 30$. Note that number of routes is 0 till $b = 1$.

and the first multi-legged route in $\bar{\mathbf{R}}$ occur at $b = 1.71$. Proposition VI.3(c) and VI.3(d) give necessary lower bounds on b for the existence of multi-legged routes in $\bar{\mathbf{R}}$ as 1.7027 and 1.692, respectively. In each case, the origin of the multi-legged route is the node $l = 0$. In Figure 2, we also see that there is a significant increase in the number of routes around $b = 2.1$. This can be explained by the fact that $b_l^* \in (2, 2.1)$ for 5 nodes. For the rest of the results we set $b = 2.5$ for which $\bar{\mathbf{R}}$ has 12219 routes and 45050 optimization variables.



(a)



(b)

Fig. 3: Simulation results for the feed-in problem. **(a):** Maximum profits as a function of total supply is in red. Maximum profits with all supply at I and their profits as a function of total supply s is in green and for different supply distributions, as a function of s, as the blue scatter points. **(b):** Number of routes utilised for obtaining maximum profits in (a).

2) Feed-In and Comparison with V.R.P.: We simulated the *feed-in* problem for a fixed demand profile with demand, d_l drawn uniformly from $[0, 250]_{\mathbb{Z}}$. The total demand was $\sum_l d_l = 2468$. Figure 3 shows the simulation results. We utilised Proposition IV.1 to generate the route set $\mathbf{R}^-$ which had 6265 routes and the resulting number of optimization variables was 12695. In Figure 3a, maximum profits for given total supply (marked in red line) is from Proposition IV.1. The maximum profits converge to the absolute maximum profits, $J_{\max} = 105033.5$ (given by Theorem IV.3). We also simulated an *equivalent macroscopic* V.R.P. by concentrating all supply at I, i.e. $S_I = s$, $S_l = 0$, $\forall l \neq I$. We observe in Figure 3a that the profits earned are far lower, compared to that of any randomly chosen supply distributions. This is explained by two elements - insufficient time-window and cost-factor. Given $T = 30$, 4 nodes do not have $r \in \mathbf{R}$ such that $o_r = I$ and $l \in V_r$. Also given $b = 2.5$, only 16 of the 23 nodes satisfy Proposition (VI.3(c)), implying at-least 7 nodes do not have $r \in \bar{\mathbf{R}}$ with $o_r = I$ and $l \in V_r$. The necessary value of b is 3.06 for all nodes to satisfy Proposition VI.3(c).

In Figure 3b we also see the number of routes used to generate maximum profits generally increases with s though after a point the number of routes used starts to reduce. We imposed the added restriction that the supply distribution is chosen from the set (27) for using in the construction of an equivalent feed-out problem.

3) Equivalence of Feed-Out and Feed-In Supply Optimization: We use the directed graph in Figure 1 and Assumptions **(A3)-(A5)** to generate a feed-out problem for the feed-in supply optimization problem. Using route set $\hat{\mathbf{R}}^-$, we generate the optimal profits for the same instances of total supply as before and compared it with the maximum profits for the feed-in supply optimization. The absolute error is in the range of 10^{-4} while the maximum relative error is 5.66×10^{-6}, which is within numerical tolerance given the solver precision is 10^{-8} and the number of variables are 12695. This verifies the equivalence of the two problems.

VIII. Conclusions

In this paper, we proposed a problem of *one-shot* coordination of first mode feed-in service, where in an operator seeks to maximize its profits with routing and allocation, to transport a known demand to a common destination on a network in a given fixed time window. We solved the problem in a macroscopic setting where we considered all supplies and demands as volumes. Using K.K.T. analysis we were able to design an offline (supply and demand independent) method that reduces the complexity of the online (after supply and demand are revealed) optimization. Then, we considered the feed-in supply optimization problem, analysed its properties and computed the absolute maximum profits that the operator can earn over all possible supply distributions for a given demand distribution. We showed an equivalence between the feed-in supply optimization problem and the *one-shot feed-out* problem, wherein the operator needs to drop-off people to their destinations from a common origin within a fixed time window. This allows us to directly apply the results and algorithms developed for the feed-in problem. Finally we analysed the limitations of a macroscopic V.R.P. in addressing the first or last-mile connectivity. In particular, a traditional V.R.P. may not be an ideal last-mile connectivity solution and a mix of multi-origin transportation model may be more viable.

Future work includes extension to a multiple time window problem, load balancing of the supply in accordance with the anticipated demand using the insights from the feed-in supply optimization problem, extension to the scenario with uncertainty about supply and demand and finally an integrated coordination of multiple modes of transportation.

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Appendix

A. Proofs of Results on One-Shot Feed-in Problem

1) Proof of Lemma II.3: From Remark II.2, we know that for an optimal solution each price $p_r^i(l) = \bar{p}_r^i(l)$. Given this, we prove the result by contradiction. Thus, let us assume $\exists (r,i,l)$ s.t. $\tilde{f}_r^i(l) < f_r^i(l)$ for an optimum solution. We have two cases:

(i) $\tilde{F}_l = F_l \leq d_l$, which implies $\sum_{r,i} \tilde{f}_r^i(l) = \sum_{r,i} f_r^i(l)$. Thus, if $\exists (r,i,l)$ s.t. $\tilde{f}_r^i(l) < f_r^i(l)$ then there must be some other (r_1, i_1, l_1) for which (5a) or the non-negativity constraint is violated. Thus, for every (r,i,l), we have $\tilde{f}_r^i(l) = f_r^i(l)$.

(ii) $\tilde{F}_l = d_l < F_l$. In this case, profits can only increase by ensuring $F_l = d_l$ as the earnings are unaffected and cost is reduced. Thus, for any optimal solution we must have $\tilde{F}_l = F_l \leq d_l$, and thus $\tilde{f}_r^i(l) = f_r^i(l)$ for every (r,i,l). □

2) Proof of Lemma III.1: We will prove this by contradiction. Let us assume $\exists r \in \mathbf{R}$, with $\theta_r \geq 2$, and $i \geq 2$ such that (12) doesn't hold for some optimal solution. Then, $\delta := f_r^* - \sum f_r^i(l)^* > 0$ units of flow costs δc_r^i units of money while earning 0 revenue.

Now consider another solution with two routes r and $\bar{r}$, which follows the same sequence of nodes as r but without the i^{th} leg of r. Let $f_r = f_r^* - \delta$ and $f_{\bar{r}} = f_{\bar{r}}^* + \delta$. Now in this solution (12) is followed strictly. Such a solution is feasible and does not lose δc_r^i while earning same revenue. Thus, this solution earns more profit than the optimal which is a contradiction. □

3) Proof of Proposition III.2: We first introduce the Lagrangian $\mathcal{L}$ for the problem (10),

$$\mathcal{L} = \bar{J} + \sum_{l \in V} \mu_l \Big(\sum_{r,i} f_r^i(l) - d_l\Big) + \sum_{r,i} \lambda_r^i \Big(\sum_{l \in V_r^i} f_r^i(l) - f_r\Big) + \sum_{l \in V} \gamma_l \Big(\sum_{r | o_r = l} f_r - S_l\Big) - \sum_r \delta_r f_r - \sum_{r,l,i} \delta_r^i(l) f_r^i(l), \quad (34)$$

where $\mu_l, \lambda_r^i, \gamma_l, \delta_r, \delta_r^i(l) \leq 0$ are the KKT multipliers. The stationarity and complementary slackness conditions are

$$\frac{\partial \mathcal{L}}{\partial f_r^i(l)} = \beta_r^i(l) + \mu_l + \lambda_r^i - \delta_r^i(l) = 0 \quad (35a)$$

$$\frac{\partial \mathcal{L}}{\partial f_r} = -c_r - \sum_i \lambda_r^i + \gamma_{o_r} - \delta_r = 0 \quad (35b)$$

$$\mu_l \Big(\sum_{r,i} f_r^i(l) - d_l\Big) = 0, \quad \lambda_r^i \Big(\sum_{l \in V_r^i} f_r^i(l) - f_r\Big) = 0 \quad (35c)$$

$$\gamma_l \Big(\sum_{r | o_r = l} f_r - S_l\Big) = 0, \quad \delta_r^i(l) f_r^i(l) = 0, \quad \delta_r f_r = 0. \quad (35d)$$

(a): In an optimal solution, if $f_r^* > 0$ then $\delta_r^* = 0$. Further as $\gamma_{o_r}^* \leq 0$ and $c_r > 0$, we can use (35b) to obtain

$$\sum_{i=1}^{\theta_r} {\lambda_r^i}^* = -c_r + \gamma_{o_r}^* < 0. \quad (36)$$

Thus, we see from (35c) that $\sum_{l \in V_r^i} f_r^i(l)^* = f_r^*$ for at least one leg i in the route r. Hence there must exist $f_r^i(l)^* > 0$ for some $i \in [1, \theta_r]_{\mathbb{Z}}$ and $l \in V_r^i$. Now, if $f_r^i(l)^* > 0$ then $\delta_r^i(l) = 0$. Hence condition (35a) implies

$$\beta_r^i(l) = -{\lambda_r^i}^* - \mu_l^* \geq -{\lambda_r^i}^* \geq 0, \text{ if } f_r^i(l)^* > 0. \quad (37)$$

(b): Now, notice that $f_r^i(l)^* \beta_r^i(l) \geq -f_r^i(l)^* {\lambda_r^i}^*$ for each (r,i,l), since if if $f_r^i(l) = 0$ then the inequality holds trivially and if $f_r^i(l) > 0$ then the condition (37) holds. Further, (36) states that $c_r = -\sum_{i=1}^{\theta_r} \lambda_r^{i\,*} + \gamma_{o_r}^*$. Thus,

$$\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \beta_r^i(l) \geq -\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \lambda_r^{i\,*},$$
$$-f_r^* \left(\sum_{i=1}^{\theta_r} \lambda_r^{i\,*} \right) \geq f_r^* c_r. \quad (38)$$

Now, using Lemma III.1, we obtain

$$\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \beta_r^i(l) \geq -\sum_{l \in V_r^1} f_r^1(l)^* \lambda_r^{1\,*} - f_r^* \sum_{i=2}^{\theta_r} \lambda_r^{i\,*}.$$

From the K.K.T. conditions, we either have $\lambda_r^{1\,*} = 0$ or $\sum_{l \in V_r^1} f_r^1(l)^* = f_r^*$. In either case using (38),

$$\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \beta_r^i(l) \geq -f_r^* \sum_{i=1}^{\theta_r} \lambda_r^{i\,*} \geq f_r^* c_r.$$

(c): Due to Lemma III.1, if $f_r^* > 0$ then $\forall\, i > 1, \exists\, l \in V_r^i$ such that $f_r^i(l)^* > 0$. In any feasible solution, for $i > 1$, if $\beta_r^i(l) < c_r^i$ and $f_r^i(l) > 0$ then we can construct another feasible solution in which $f_r^i(l) = 0$ keeping all other node allocation variables unchanged. The value of the objective is *strictly* more with such a new solution. Thus, $\beta_r^i(l) \geq c_r^i$ if $f_r^i(l)^* > 0$ for $i > 1$ in an optimal solution of Problem (10).

(d): In part (b), by setting $\theta_r = 1$, we get

$$\sum_{l \in V_r^1} f_r^1(l)^* \beta_r^1(l) \geq f_r^* c_r.$$

Similarly, setting $\theta_r = 1$ in (36) gives us $\lambda_r^1 \leq -c_r < 0$, which along with (35c) means that in any optimal solution, constraint (6a) is a strict equality for single legged routes. Therefore, we have $\sum_{l \in V_r^1} (\beta_r^1(l) - c_r) f_r^1(l)^* \geq 0$. Now, part (d) follows in the same way as part (c). ☐

B. Proofs of Results on Feed-in Supply Optimization

1) Proof of Proposition IV.1: We prove all parts except **(a)** by contradiction. **(a)** is true because Theorem III.3 applies for every possible supply distribution.

(b): Let us assume there exists an optimal solution, which we denote using a superscript $*$, in which $f_r^* > 0$ for a route $r \in \bar{\mathbf{R}}$ such that $f_r^1(o_r)^* < f_r^*$. Consider. Then, there are the following two cases. **(i)** $f_r^1(o_r)^* > 0$ and $f_r^i(l)^* = 0$ for all other legs and nodes (i, l) on route r; and **(ii)** there is a pair $(i_1, l_1) \in [1, \theta_r]_{\mathbb{Z}} \times V_r$ with $(i_1, l_1) \neq (1, o_r)$ and $f_r^{i_1}(l_1)^* > 0$. Note that Lemma III.1 implies that scenario (i) may occur only if the route is simple ($\theta_r = 1$).

(i) In this case, clearly the original solution cannot be optimal because $f_r^* - f_r^1(o_r)^*$ volume of vehicles simply traverse the route without serving any demand, thus incurring a non-zero cost while earning nothing.

(ii) Without loss of generality let (i, l_1) be the leg, node pair other than $(1, o_r)$ that first appears in the sequence given by the route r such that $f_r^i(l_1)^* > 0$. Then consider the route $\bar{r}$, which is the sub-route of r formed by excluding all nodes in r that occur prior to (i, l_1) so that $o_{\bar{r}} = l_1$. Thus, for each leg and node pair (j, m) of the route $\bar{r}$ there is a unique leg and node pair (k, l) of route r such that $(j-i+1, m) = (k, l)$ and moreover they appear in the same order. Now consider a solution $f_r = f_r^1(o_r) = f_r^1(o_r)^*$ and $f_{\bar{r}} = f_{\bar{r}}^* + (f_r^* - f_r^1(o_r)^*)$ and such that $f_{\bar{r}}^j(m) + f_r^k(l) = f_{\bar{r}}^j(m)^* + f_r^k(l)^*$ for every $m \in V_{\bar{r}}^i$ and for each leg j of route $\bar{r}$. This solution is feasible and earns higher profits than the original solution as the flow $(f_r^* - f_r^1(o_r)^*)$ does not have to traverse the sequence of nodes from $(1, o_r)$ to (i, l_1) and the node allocations are unchanged. This again contradicts the assumption that the original solution is optimal. Thus, for each $r \in \bar{\mathbf{R}}$, $f_r^1(o_r)^* = f_r^*$. As a consequence of this fact and Lemma III.1, we also have (17). Also, as $d_I = 0$ therefore $f_r^* = 0$, $\forall r$ s.t. $o_r = I$.

(c): Assume an optimal solution with $f_r^* > 0$, for a $r \in \bar{\mathbf{R}}$ with $\beta_r^1(o_r) < c_r^1$. Using **(b)**, we know $f_r^1(o_r)^* = f_r^* > 0$. However, one could set $f_r^1(o_r)^* = 0$ and move the supply on o_r as in part **(b)**. Then, the so constructed solution would again earn higher profits, which contradicts the assumption that the original solution is optimal. Consequently, with Proposition III.2(c), $\exists\ l \in V_r^i,\ s.t.\ f_r^i(l)^* > 0,\ \beta_r^i(l) \geq c_r^i, \forall i \in [1, \theta_r]_{\mathbb{Z}}$.

(d): Suppose that a node l violates the property in **(A2)** and yet $f_r^i(l)^* > 0$ for some route r and a leg i. Then by part **(c)** we must have $\beta_r^i(l) - c_r^i \geq 0$. Now, for the service tuple (r, i, l), consider a simple route q, which is the sub-route of route r from the last visit to node l in leg i to I in that leg. As a result, $o_q = l$ and $q \in \mathbf{R}$. Now, notice that

$$\beta_q^1(l) - c_q^1 \geq \beta_r^i(l) - c_r^i \geq 0,$$

since q is a sub-route of the leg i of route r. This contradicts the assumption that l violates the property in **(A2)**.

(e): One can construct another route q from r by avoiding the cycle. Again, moving the supply to this route (in a manner similar to the previous parts) earns more profits.

(f): In the scenario $s \leq \sum_l d_l$, the key observation is that the full demand cannot be served by simple routes. Thus, if $\sum_{r | o_r = l} f_r^* < S_l$ for some node $l \in V$ then there is some unused supply. Such redundant supply could potentially be used to serve more demand either at node l or moved to a different node $\bar{l}$ to meet the demand there with simple routes. Assumption **(A2)** implies that there exist simple routes to which if the redundant supply is reallocated then the profits are higher. This contradicts that the original solution is optimal. Thus, in every optimal solution, we have

$$S_l = \sum_{r | o_r = l} f_r^* \leq F_l^* \leq d_l, \quad \forall l \in V,$$

where the second inequality is just one of the constraints in the optimization problem. ☐

2) Proof of Lemma IV.2: By the definition of $\mathcal{R}(l)$ in (21), $\beta_r^1(l) - c_r > \beta_q^1(l) - c_q$, $\forall\ r \in \mathcal{R}(l)$ and $\forall q \in (\sigma(l) \setminus \mathcal{R}(l))$. Every other route $\bar{r}$ either it originates from a different location or it has multiple legs and in each case the leg/route cost is higher and the operator revenues are lower. Therefore, $\exists q \in (\sigma(l) \setminus \mathcal{R}(l))$ for which $\beta_q^1(l) - c_q > \beta_{\bar{r}}^i(l) - c_{\bar{r}}^i$. Hence, $\beta_r^1(l) - c_r > \beta_{\bar{r}}^i(l) - c_{\bar{r}}^i$ for all $r \in \mathcal{R}(l)$ and $\bar{r} \notin \mathcal{R}(l)$. Further, for $\forall\ r \in \mathcal{R}(l)$ and $\forall q \in (\sigma(l) \setminus \mathcal{R}(l))$, the perceived costs satisfy

$$(\alpha t_r + c_r) - (\alpha t_q + c_q)$$

$$= (\alpha t_r + c_r - g_l + 1) - (\alpha t_q + c_q - g_l + 1) - (\beta_r^1(l) - c_r) + (\beta_q^1(l) - c_q) < 0, \quad (39)$$

where we have used (8) and (11) and the fact that $o_z = l$ for all $z \in \sigma(l)$, which implies that the route traversal time $t_z = (T - t_z^1(l))$. This proves the result. □

C. Proofs of Results on One-Shot Feed-out

1) Proof of Lemma V.3: As for both the problems the time window, best travel cost and the best travel time are same, it therefore suffices to show that $T - t_{\bar{r}}^{\theta_r - i+1}(l) = \hat{t}_r^i(l)$. Note that the i^{th} leg of r is same as the $(\theta_r - i + 1)^{\text{th}}$ leg of $\bar{r}$ in reverse. Also, $t_{\bar{r}}^{\theta_r - i+1}(l)$ is the last possible pick-up time along $(\bar{r}, \theta_r - i + 1, l)$, based on the observations of (8) which implies that $T - t_{\bar{r}}^{\theta_r - i+1}(l)$ is the first possible drop-off time for the reverse route r along the service tuple (r, i, l). □

D. Proofs of Results on Best Transportation Parameters

1) Proof of Lemma VI.2: Consider an arbitrary node $l \in V$ and a service tuple (r, i, l) such that $l \in V_r^i$. Then, from (8) and (11) notice that

$$\begin{aligned}\beta_r^i(l) - c_r^i &= g_l(b) - \alpha(T - t_r^i(l)) - c_r^i - 1 \\ &\le (\alpha t_q + bc_q) - \alpha(T - t_r^i(l)) - c_r^i - 1, \forall q \in \sigma(l) \\ &\le (b-1)c_r^i - 1,\end{aligned}$$

where the first inequality follows from the definition of $g_l(b)$ in (33) and the second inequality from letting q be the sub-route of V_r^i from the last occurrence of l to I. Thus, if $b \le 1$ then for each (r, i, l), we have $\beta_r^i(l) - c_r^i < 0$. Since this is true for all (r, i, l), we conclude that $\bar{\mathbf{R}}$ is empty. □

2) Proof of Proposition VI.3: Let condition (a) be true. Then route $r \in \mathbf{R}_2$ (see (15)), which implies that for each secondary leg of route r, $w_i \ge 0$. This implies there exists a route $\bar{r}$ equal to the leg $i > 1$ of route r. Thus, $o_{\bar{r}} = I$, $\theta_{\bar{r}} = 1$ and $w_{\bar{r}}^1 \ge 0$, which implies $\bar{r} \in \bar{\mathbf{R}}$. Therefore, (a) ⇒ (b). Now, if condition (b) is true then $\exists r \in \mathbf{R}_1$ (see (14)), that is, $w_r^1 = \max_{l \in V_r^1}\{\beta_r^1(l) - c_r\} \ge 0$. Hence, from (8) and (11),

$$g_l(b) \ge \alpha(T - t_r^1(l)) + c_r + 1, \text{ for some } l \in V_r, \; l \neq I,$$

where we have used the fact that $g_I(b) = 0$ for all $b \ge 0$ and hence $l \neq I$. Now, note that

$$\begin{aligned}\alpha(T - t_r^1(l)) + c_r &\ge \alpha(T - t_r^1(l)) + c^*(I, l) + c_r(l, I) \\ &\ge g_l(1) + c^*(I, l),\end{aligned}$$

where we have split the route cost c_r into $c_r(I, l)$, the cost to go from I to l on route r, and $c_r(l, I)$, the cost to go back from l to I, and lower bounded $c_r(I, l)$ by $c^*(I, l)$, the optimal cost to go from I to l. Therefore, $g_l(b) \ge g_l(1) + c^*(I, l) + 1$, thus proving (b) ⇒ (c).

Now, suppose condition (c) is true for $l \in V$, $l \neq I$. Now using (33), we get $g_l(b) = \alpha t_{r(l,b)^*} + bc_{r(l,b)^*}$ and $g_l(1) = \alpha t_{r(l,1)^*} + c_{r(l,1)^*}$. Therefore, (c) can be expressed as

$$b \ge \left(\frac{1 + c^*(I, l) + \alpha(t_{r(l,1)^*} - t_{r(l,b)^*}) + c_{r(l,1)^*}}{c_{r(l,b)^*}} \right). \quad (40)$$

Now, notice from (32) that $c_{r(l,1)^*} \ge c_{r(l,b)^*}$ and $t_{r(l,b)^*} \le t_{r(l,\infty)^*}$. Hence, VI.3(d) holds and as a result, (c) ⇒ (d).

Now, suppose that $\exists l \neq I$ such that (c) is satisfied. Consider the route r which is constructed by stitching the cheapest path to go from I to l with cost $c^*(I, l)$ and a path $r(l, 1)^*$ with perceived cost $g_l(1)$, to go from l to I. Then, route $r \in \mathbf{R}$ for sufficiently large T. Now observe that

$$\begin{aligned}\beta_r^1(l) - c_r &= g_l(b) - \alpha(T - t_r^i(l)) - 1 - c^*(I, l) - c_{r(l,1)^*} \\ &= g_l(b) - \alpha t_{r(l,1)^*} - c_{r(l,1)^*} - c^*(I, l) - 1 \\ &= g_l(b) - g_l(1) - c^*(I, l) - 1 \ge 0,\end{aligned}$$

where the first equality follows from (11) and (8) and the construction of the route to have the specific route cost, the second equality again follows due to the fact that the specific construction of the route from l to I gives the traversal time as $t_{r(l,1)^*}$ (see Remark VI.1) and in the third equality, we have used the definition of $g_l(1)$. The inequality follows from the assumption that (c) is true. Hence, $r \in \mathbf{R}_1 \subset \bar{\mathbf{R}}$ by (14) and we conclude that for sufficiently large T (c) ⇒ (b). □