Evolution of a Population of Selfish Agents on a Network ⋆

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Abstract: In this work, we consider a population composed of a continuum of agents that seek to selfishly minimize a cost function by moving on a network. The nodes in the network may represent physical locations or abstract choices. Taking inspiration from how water distributes itself in a system of connected tanks of varying heights, we formulate a best response dynamics for the population. In this dynamics, the population in each node simultaneously seeks to redistribute itself according to the 'best response' to the state of the population in the node's neighborhood. We provide an algorithm to determine the best response as a function of the state of the population. We then show that given the state of the population, the best response is unique. For the continuous time version of the best response dynamics, we show asymptotic convergence to an equilibrium point for an arbitrary initial condition. We then explore a second dynamics, in which the population evolves according to centralized gradient descent of the social cost. Again, we show asymptotic convergence for an arbitrary initial condition. We illustrate our results through simulations.

Keywords: Multi-agent systems, population dynamics, best response dynamics, evolution on networks

1. INTRODUCTION

Large scale *multi-agent systems*, including those with selfish agents, have been of interest to the scientific community for several years. A number of frameworks have emerged over the years to explore various facets of such systems. Population games and evolutionary dynamics, opinion dynamics and swarm control are a few examples. In such large scale systems, the primary interest is in the evolution of the population as a whole rather than that of specific, individual agents. In this paper, we seek to model the evolution of a population of selfish and myopic agents that may move on a network.

1.1 Literature Survey

Population games and evolutionary dynamics (Sandholm, 2010) are well established areas that explore such large scale systems composed of selfish agents. However, in much of the work in this literature, there is no state dependent restriction on the available actions as a network might impose. Evolutionary dynamics on a graph (Lieberman et al., 2005; Pattni et al., 2015; Allen and Nowak, 2014) is concerned with a finite number of agents modeled as nodes with the graph being a represention of the interactions between different agents. References (Barreiro-Gomez et al., 2016; Zino et al., 2017; Barreiro-Gomez and Tembine, 2018) have a formulation similar to the current work but also have significant differences. Zino et al. (2017) consider the underlying graph to be a complete graph wherein every agent can interact with every other agent albeit with different frequency rates. On the other hand, (Barreiro-Gomez et al., 2016; Barreiro-Gomez and Tembine, 2018) consider the underlying graph in full generality but give local results where the initial condition and the Nash Equilibrium can only be in the relative interior of the n-dimensional probability simplex.

Another relevant area is that of opinion dynamics, which has its roots in sociology and seeks to construct dynamical models of the evolution of social behavior as represented by the opinions of agents. Although, by now, the literature in this area is vast (Proskurnikov and Tempo, 2017, 2018), almost all the work is in the context of opinions evolving in a continuous space and for finitely many agents, perhaps with the exception of (Hendrickx and Olshevsky, 2016) which considers a continuum of agents.

Yet another related area is that of swarms and swarm control. Krishnan and Martínez (2018) consider the swarm to be a continuum evolving in a continuous space and use a partial differential equations framework for designing control algorithms. Another established way of modelling a swarm is to discretize the space and consider the agents to be moving from one cell to another. Most of these works consider the movement of the agents in the swarm to be probabalistic and hence turn to Markov chains (Chattopadhyay and Ray, 2009; Açıkmeşe and Bayard, 2015; Bandyopadhyay et al., 2013). While some use the convergence properties of markov chains in order to design appropriate control actions for the swarm to reach a desired distribution (Açıkmeşe and Bayard, 2015), others focus on designing the markov chains so that the swarm converges to the stationary distribution (Chattopadhyay and Ray,

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2009). Another way to look at swarms is distributed coverage control (Diaz-Mercado et al., 2015). Finally, Wang and Li (2017, 2018) explore the problem of optimal control of an ensemble of bilinear systems.

1.2 Contributions

In this paper, we study the evolution of a *population* of selfish and myopic *agents* on a network. We model the population as a continuum of agents, with different fractions of the population located on different nodes in the network. The nodes in the network may represent physical locations or *choices*, in a more abstract sense, available to the infinitesimal agents. Each agent repeatedly seeks to minimize a cost function by moving/revising from its current node/choice to one of the neighboring nodes/choices in the network. An agent may also choose not to revise its choice at a given time. Thus, at each time instant, the network imposes constraints on the set of choices that an agent can revise to. Under this setup, we model the evolution of the population on the network starting from an arbitrary initial configuration. For this, we take inspiration from nature - in particular from how water in a system of connected tanks redistributes itself from an initial configuration. Specifically, we model the evolution of the population based on the best response dynamics. We also consider the centralized network restricted potential minimization dynamics. In each case, we demonstrate analytically that for all initial conditions, the trajectories converge to an equilibrium point. We compare the two dynamics through simulations.

Compared to (Barreiro-Gomez et al., 2016; Zino et al., 2017; Barreiro-Gomez and Tembine, 2018), in this work, we consider an arbitrary graph and model the evolution of the population on the simplex of dimension equal to the number of nodes in the graph. We allow both the initial condition of the population and the equilibrium of the dynamics to be present anywhere on the simplex. Compared to the opinion dynamics literature, our proposed model has several distinctive features - a continuum of agents, discrete space for the opinions (choices) interrelated by a network and most significantly, the dynamics being the result of the agents optimizing a cost or utility function, as in game theoretic evolutionary dynamics. The literature on swarm control has its roots in robotics and hence the dynamics of the agents may be designed. In this paper, however, we assume that the dynamics is the result of agents' inherent selfish and myopic nature.

1.3 Organization

The rest of the paper is organized as follows. In Section 2, we provide the basic framework and setup the overall problem that we address. In Section 3, we give a dynamics in the spirit of best response dynamics in game theory and analyze its convergence. In Section 4, we give a centralized dynamics and again analyze its convergence. In Section 5, we illustrate the two dynamics and our analytical results through simulations. In Section 6 we summarize the paper and provide directions for further extensions.

1.4 Notation and Definitions

By $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{Z}$ we denote the sets of real numbers, nonnegative real numbers and integers respectively. We let $[p, q]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, p \leq x \leq q\}$. $\mathbb{R}^n$ (similarly $\mathbb{R}^n_+$) is the cartesian product of $\mathbb{R}$ (equivalently $\mathbb{R}_+$) with itself n times. If $v \in \mathbb{R}^n$, we denote the i^{th} component of v by v_i. The support of the vector is defined as $supp(v) := \{i \in [1, n]_{\mathbb{Z}} \,|\, v_i \neq 0\}$. We let $\mathbb{1}$ to be the vector, of appropriate size, with all its components as 1. The empty set is denoted by $\varnothing$. If $\mathcal{Q}$ is an ordered countable set, then $\mathcal{Q}_i$ denotes the i^{th} member of $\mathcal{Q}$ and $|\mathcal{Q}|$ is used to represent the cardinality of $\mathcal{Q}$. (i, j) is used to denote an ordered pair. If $v \in \mathbb{R}^n$, $v \geq 0$ is used to denote term wise inequalities. For a function $f(x) : \mathbb{R}^n \to \mathbb{R}$, $\nabla_x f$ is used to denote the gradient of f with respect to x, i.e., the j^{th} component of $\nabla_x f$ is $\frac{\partial f}{\partial x_j}$. We denote by $\mathcal{S}^n$, the n-dimensional simplex $\mathcal{S}^n := \{v \in \mathbb{R}^n \mid v \geq 0 \;,\; \mathbb{1}^T v = 1\}$.

2. FRAMEWORK AND PROBLEM SETUP

In this paper, we consider a *population* composed of a continuum of *agents* that seek to selfishly minimize a cost function by moving on a network or a graph. Let $\mathcal{V}$ be a set of nodes, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ be a set of edges and $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be an undirected graph that does not contain any self loops or multiple edges. The nodes in the network represent either physical locations or they may be *choices*, in a more abstract sense, that are available to the infinitesimal *agents* constituting the population. Let $N := |\mathcal{V}|$ be the total number of nodes and $M := |\mathcal{E}|$ be the total number of edges in the graph. Let $x_i \in [0, 1]$ be the *fraction* of the population in node i, or equivalently making the choice i. We assume that the overall population is fixed and, without loss of generality, assume that $\sum_{i \in \mathcal{V}} x_i = 1$. The cost that the fraction x_i incurs as a whole is

$$p(a_i, x_i) := a_i\, x_i + \frac{1}{2} x_i^2, \tag{1}$$

where $a_i \in \mathbb{R}$ is the *node parameter* associated with node i. For example, $-a_i$ can be thought of as a *measure of profitability* of node i.

Let $x \in \mathcal{S}^N \subset \mathbb{R}^N$ and $a \in \mathbb{R}^N$ be the vectors with x_i and a_i as the i^{th} component, respectively. In this paper, we are interested in the evolution of x, the fractions of the population making each choice $i \in \mathcal{V}$. We see that the underlying graph $\mathcal{G}$ constrains the transition of choices of the infinitesimal agents. For example, a change in the choice of an infinitesimal "agent" or fraction of agents from $i \in \mathcal{V}$ to $j \in \mathcal{V}$ must occur only through a sequence of changes corresponding to some path from i to j in $\mathcal{G}$.

Remark 1. (Water and Connected Water Tanks Analogy). As shown in Figure 1, we interpret each node i as a water tank of unit cross-sectional area, with its base at a height a_i. Then, x_i represents the volume of water in the tank. Notice that the potential energy of the water in tank i is precisely as in (1), assuming the density of water and acceleration due to gravity each is 1 unit. We assume that each tank has the capacity to hold the entire volume of water (entire population). The graph $\mathcal{G}$ describes the network of connections between the water tanks. Given an arbitrary $a \in \mathbb{R}^N$ and an initial $x \in \mathcal{S}^N$, we suppose that each water particle (infinitesimal agent) seeks to minimize its potential energy (cost). •

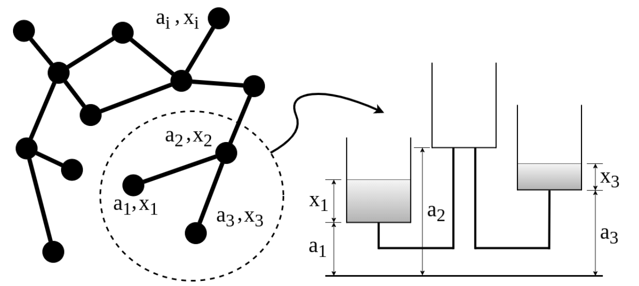


Fig. 1. Graph structure and the connected water tanks interpretation.

2.1 Dynamics and Problem Statement

Note that we are only interested in the evolution of the population fractions x_i and not in the evolution of each infinitesimal agent per se. Thus, we suppose that at each time instant, the fraction in node i seeks to selfishly minimize its overall cost by redistributing itself among node i and its neighbors while assuming that the fraction in the neighboring nodes does not change. The overall dynamics is the resultant of the fractions x_i in each node i simultaneously seeking to selfishly and myopically minimize their potential energy, in a manner constrained by the graph $\mathcal{G}$. We also consider an evolution of x where the decision making is centralized. In such a scenario, the fraction of population in node i still seeks to myopically minimize its overall cost by redistributing itself among node i and its neighbors, but now with the knowledge and coordination of the redistributing choices of the fractions in the other nodes as well.

Both the evolution of the population and the steady state (if it exists) are significantly affected by the graph $\mathcal{G}$ and the node parameters a. As a first step, we seek to capture the above notional dynamics as a discrete-time or continuous-time dynamical system, with the dependence on the node parameters a and the graph $\mathcal{G}$ suitably captured as

$$x(t+1) = f^d(x(t), a, \mathcal{G}), \quad \text{or} \quad \dot{x} = f^d(x, a, \mathcal{G}).$$

In this paper, we are interested in the case with a constant a and seek to characterize the dynamics and its convergence properties.

3. SIMULTANEOUS BEST RESPONSE DYNAMICS

In this section, we propose an initial characterization of the best response dynamics and analyze properties of its solutions, chiefly its convergence. Here, each of the fractions of population take a decision to redistribute themselves among their neighboring nodes with local information about the overall population distribution. First we give an algorithm to determine the best response and formulate the dynamics. Then, we analyze the properties of its solutions, including the convergence property.

3.1 Best Response

Let $\mathcal{N}^i$ be the set of all neighbors of node i in the graph $\mathcal{G}$ and let $\overline{\mathcal{N}}^i = \mathcal{N}^i \cup \{i\}$. Given the undirected graph $\mathcal{G}$, let $\mathcal{A} := \bigcup_{\{i,j\}\in\mathcal{E}}\{(i,j),(j,i)\}$. It is easy to see that $|\mathcal{A}| = 2|\mathcal{E}| = 2M$. The arcs in $\mathcal{A}$ are used to model the inflow and outflow of the population between a given node and its adjacent nodes. Given a configuration of x and a, the population fraction in a node i may be able to reduce its cost by redistributing itself among node i and its neighboring nodes $\mathcal{N}^i$. Let δ_{ij} denote the *outflow*, that is the fraction of population that moves from node i to node $j \in \mathcal{N}^i$ through the arc (i,j). The fraction x_i determines the optimal δ_{ij} that would minimize its own overall cost under the assumption that the fractions in other nodes do not redistribute. However, the overall dynamics is the resultant of each fraction x_i simultaneously redistributing itself in the above "myopic" sense. This behaviour is in the spirit of best response dynamics (Sandholm, 2010) in evolutionary dynamics.

Thus the outflows δ_{ij} from node i are determined by solving the following optimization problem

$$\begin{aligned} &\mathbf{P}_1(i): &(2)\\ &\min_{\{\delta_{ij}\,|\,j\in\overline{\mathcal{N}}^i\}} \sum_{j\in\mathcal{N}^i} p(a_j + x_j, \delta_{ij}) + p(a_i, \delta_{ii})\\ &\text{s.t. } \delta_{ii} + \sum_{j\in\mathcal{N}^i} \delta_{ij} = x_i, \quad \delta_{ij} \geq 0,\ \forall j \in \overline{\mathcal{N}}^i. \end{aligned}$$

The non-negativity constraints $\delta_{ij} \geq 0$ and $\delta_{ii} \geq 0$ ensure that the outflows to the neighboring nodes and the fraction that remains in node i each is non-negative. Notice that

$$p(a_j, x_j + \delta_{ij}) = a_j x_j + \frac{1}{2}x_j^2 + p(a_j + x_j, \delta_{ij}).$$

Hence, with δ_{ij} as the optimization variables, one can also interpret the objective in the problem $\mathbf{P}_1(i)$ as one of minimizing the potential energy of the fraction in node i and its neighbors with the optimization variables still as the outflows from node i to its neighbors.

3.2 An Algorithm to Compute the Best Response

The problem in (2) is a quadratic program (Boyd and Vandenberghe, 2004) and is also convex in $\{\delta_{ij}\}_{j\in\overline{\mathcal{N}}^i}$. The cost function, in particular, is also strictly convex. Moreover, if $x_i \in \mathbb{R}_+$, then the problem $\mathbf{P}_1(i)$ is always feasible since $\delta_{ij} = 0$ for all $j \in \overline{\mathcal{N}}^i$ is feasible. Hence the problem $\mathbf{P}_1(i)$ possesses a unique minimizer. Algorithm 1 gives a procedure to compute the unique minimizer, $\{\delta^*_{ij}\}_{j\in\overline{\mathcal{N}}^i}$, of $\mathbf{P}_1(i)$ given a, x and the graph $\mathcal{G}$.

Algorithm 1 relies on the observation that the fraction in node i perceives $h_{ij} := a_j + x_j$ as the node parameter for each $j \in \mathcal{N}^i$. Hence, we call h_{ij} as the pseudo cost parameter. Algorithm 1 is best understood in the context of Remark 1 with nodes interpreted as tanks, population fractions x_i interpreted as the volume of water in tank i and δ_{ij} as the outflow of water from tank i to tank j. In this context, h_{ij} is the pseudo height of the tank j that is perceived by the volume of water x_i in node i. Then the objective in (2) is simply the minimization of the potential energy of the volume of water x_i through its redistribution among tank i and its neighbors at pseudo heights h_{ij}. In the problem $\mathbf{P}_1(i)$, we let the pseudo height of tank i be $h_{ii} := a_i$. The main idea of the algorithm is based on the following observations.

Remark 2. (Observations about the unique minimizer of problem $\mathbf{P}_1(i)$). The unique minimizer $\{\delta^*_{ij}\}_{j\in\overline{\mathcal{N}}^i}$ to the problem $\mathbf{P}_1(i)$ satisfies the following two properties.

- If $\delta^*_{ij} > 0$ for some $j \in \overline{\mathcal{N}}^i$ then $\delta^*_{iq} > 0$ for all q such that $h_{iq} \leq h_{ij}$.
- If $\mathcal{D}$ is the set of all nodes $j \in \overline{\mathcal{N}}^i$ for which $\delta^*_{ij} > 0$ then there exists a constant H such that

$$h_{ij} + \delta^*_{ij} = H, \quad \forall j \in \mathcal{D}.$$

For each feasible solution that does not satisfy these conditions one can construct another feasible solution that has a lower cost. •

The main iterations in Algorithm 1 compute the set $\mathcal{D}$ and the value of H.

Algorithm 1: Find minimizer of $\mathbf{P}_1(i)$

Input: a_j, x_j, $\forall j \in \overline{\mathcal{N}}^i$
Output: δ^*_{ij}, $\forall j \in \overline{\mathcal{N}}^i$

1: $h_{ij} \leftarrow a_j + x_j$, $\forall j \in \mathcal{N}^i$ {pseudo height for neighbors of i}
2: $h_{ii} \leftarrow a_i$ {pseudo height for node i}
3: **if** $x_i = 0$ **then**
4: $\quad h \leftarrow -\infty$ {initial height if $x_i = 0$}
5: **else if** $x_i > 0$ **then**
6: $\quad h \leftarrow a_i + x_i$ {initial height if $x_i > 0$}
7: **end if**
8: $\mathcal{M} \leftarrow \varnothing$ {set of nodes with $\delta^*_{ij} > 0$}
9: $\overline{\mathcal{M}} \leftarrow \{j \in \overline{\mathcal{N}}^i \mid h_{ij} < h\}$ {candidate nodes for $\delta^*_{ij} > 0$}
10: **while** $(\overline{\mathcal{M}} \setminus \mathcal{M}) \neq \varnothing$ **do**
11: $\quad \mathcal{M} \leftarrow \mathcal{M} \cup \underset{(\overline{\mathcal{M}} \setminus \mathcal{M})}{\operatorname{argmin}}\{h_{ij}\}$
12: $\quad h \leftarrow \dfrac{\left(\sum_{k \in \mathcal{M}} h_{ik}\right) + x_i}{|\mathcal{M}|}$
13: $\quad \overline{\mathcal{M}} \leftarrow \{j \in \overline{\mathcal{M}} \mid h_{ij} < h\}$
14: **end while**
15: $\delta^*_{ij} \leftarrow h - h_{ij}$, $\forall j \in \mathcal{M}$
16: $\delta^*_{ij} \leftarrow 0$, $\forall j \in \overline{\mathcal{N}}^i \setminus \mathcal{M}$

Theorem 3. If $x_i \in \mathbb{R}_+$ then Algorithm 1 computes the unique optimizer of problem $\mathbf{P}_1(i)$ in (2). □

We omit the proof this theorem due to space constraints. Its main ideas are to show that the unique minimizer must indeed satisfy the properties in Remark 2 and to demonstrate that H and the set $\mathcal{D}$ are in fact h and $\mathcal{M}$ returned by Algorithm 1.

3.3 Discrete-Time and Continuous-Time Versions of the Dynamics

Algorithm 1 can be repeated for all $i \in \mathcal{V}$ to get the set $\{\delta^*_{ij}\}_{(i,j)\in\mathcal{A}}$ of all outflows on every arc $(i,j) \in \mathcal{A}$ and $\{\delta^*_{ii}\}_{i\in\mathcal{V}}$ the fraction of population that is retained in node i in the problem (2). This gives the best response of the population x_i in each node $i \in \mathcal{V}$. Moreover, Algorithm 1 can also be used to justify Remark 4.

Remark 4. (Best response outflows are not bi-directional). For each $j \in \mathcal{N}^i$, if $\delta^*_{ij} > 0$ then $\delta^*_{ji} = 0$. Thus $\delta^*_{ij}\delta^*_{ji} = 0$ $\forall j \in \mathcal{N}^i$. The reasoning for this observation is that if $\delta^*_{ij} > 0$ for $j \in \mathcal{N}^i$ then $a_j + x_j = h_{ij} < a_i + x_i = h_{ji}$. Thus, we see that in Step 9 of Algorithm 1 when implemented for $\mathbf{P}_1(j)$, node $i \notin \overline{\mathcal{M}}$ and hence $i \notin \mathcal{M}$ of $\mathbf{P}_1(j)$. •

Using this observation, and the fact that $\delta^*_{ii} = x_i - \sum_{j\in\mathcal{N}^i} \delta^*_{ij}$, which follows from the constraints in (2), we can write the evolution of x, according to the simultaneous best response dynamics, as

$$D_1(s) \;:\; x(t+1) = x(t) + s\,A\;\Delta^*(a, x(t)). \tag{3}$$

where $s \in (0, 1]$ is a step size parameter, which captures how frequently the population fraction in a node i becomes aware of the changed population distribution x compared to the rate at which the population redistribution occurs.

In (3), $A \in \mathbb{R}^{N\times 2M}$ is the incidence matrix of the graph which can be formed using $\mathcal{V}$ and $\mathcal{A}$. We first number each arc in $\mathcal{A}$ and let $\mathcal{A}_n$ be the n^{th} arc, with $n \in [1\ ,\ 2M]_{\mathbb{Z}}$. If $\mathcal{A}_n = (i,j)$, then $A_{in} = -1, A_{jn} = 1$ and $A_{kn} = 0$, $\forall\, k \in \mathcal{V}\backslash\{i,j\}$. Similarly, we assemble the elements of the set $\{\delta^*_{ij}\}_{(i,j)\in\mathcal{A}}$ into the vector $\Delta^* \in \mathbb{R}^{2M}$ as

$$\Delta^*_n := \delta^*_{ij}, \text{ for } n \in [1\ ,\ 2M]_{\mathbb{Z}} \text{ s.t. } \mathcal{A}_n = (i,j). \tag{4}$$

It should be noted that if either a or x is changed, the set of optimizers $\{\delta^*_{ij}\}_{(i,j)\in\mathcal{A}}$ and hence Δ^* will change. Thus $\Delta^*(a,x)$ in (3) is written as a function of a and x.

We will refer to the dynamics in (3) as the best response dynamics (BRD). In the limit that $s \to 0$, the dynamics in the difference equation "approaches" that of the differential equation

$$\dot{x} = A\,\Delta^*(a,x). \tag{5}$$

Lemma 5. (Existence and Uniqueness of Solutions for Continuous-Time BRD). Let $a \in \mathbb{R}^N$ be fixed. The state equation in (5) with an initial condition $x(0) \in \mathcal{S}^N$ has a unique solution $\forall\, t \geq 0$. □

We skip the proof of the lemma due to space constraints. The main idea for the proof is to show that $\Delta^*(a,x)$ is locally Lipschitz in x and that the trajectories of (5) are confined to the simplex, which is a compact set.

3.4 On the Convergence of Continuous-time BRD

Here we introduce the function,

$$V(a,x) := \sum_{i\in\mathcal{V}} p(a_i, x_i), \tag{6}$$

which is the aggregate potential energy of the population as a whole. Note that $V(a,x)$ is a strictly convex function in x. For simplicity, we skip the first argument of V and write it as only a function of the population distribution, i.e. $V(x)$, wherever there is no confusion.

The set of equilibrium points $\mathcal{X}(a)$ of the dynamics (5), as a function of a is given by

$$\mathcal{X}(a) := \big\{\, x \in \mathcal{S}^N \;\big|\; a_i + x_i \leq a_j + x_j \quad \forall\, j \in \mathcal{N}^i, \forall\, i \in \mathit{supp}(x) \big\} \tag{7}$$

In the following theorem, we show that for each initial $x(0)$, the state converges to some equilibrium point in $\mathcal{X}(a)$. We omit the proof due to space limitations. The main idea of the proof is based on the observations in Remark 7 and LaSalle's invariance theorem.

Theorem 6. (Continuous-time BRD converges asymptotically to an equilibrium point). For a fixed $a \in \mathbb{R}^N$, let the evolution of x be governed by the dynamics in (5) with an initial condition $x(0) \in \mathcal{S}^N$. Then $\lim_{t\to\infty} x(t) = \overline{x}$, where $\overline{x} \in \mathcal{X}(a)$, with $\mathcal{X}(a)$ defined in (7). □

Remark 7. (Two Interpretations of the Rate of Change of Potential Energy for Continuous-Time BRD). The derivative of V along the trajectories of (5), i.e. $\dot{V} = (\nabla_x V)^T \dot{x}$ can be written as

$$\begin{aligned} \dot{V} &= (\nabla_x V)^T A\,\Delta^* = (\Delta^*)^T A^T \nabla_x V \\ &= \sum_{(i,j)\in\mathcal{A}} \delta^*_{ij} \left(\frac{\partial V}{\partial x_j} - \frac{\partial V}{\partial x_i} \right) \\ &= \sum_{(i,j)\in\mathcal{A}} \delta^*_{ij} \left[(a_j + x_j) - (a_i + x_i) \right]. \end{aligned} \tag{8}$$

In (8), two different ways of looking at the rate of change of potential energy along the trajectories of (5) is provided. In the first line, we are directly computing the rate of change of potential energy of the fraction in each node and summing them up. In the second and the third lines, we are instead computing the change in the potential energy due to each individual inter-nodal outflows and then adding them up, This alternate perspective makes the computation and bounding of the rate of change of potential energy particularly easy. •

4. NETWORK RESTRICTED POTENTIAL MINIMIZATION

In this section, we analyze the centralized or socially optimal dynamics given by gradient descent of the potential energy, though restricted by the network or the graph. Again, we let a remain constant and let x evolve on its own. We use V defined in (6) to describe the potential energy of the system. We let the set of arcs $\mathcal{A}$ and the incidence matrix A be defined as in the text preceding (4) in Section 3. Let the elements of the set $\{\delta_{ij}\}_{(i,j)\in\mathcal{A}}$ be assembled into the vector $\Delta \in \mathbb{R}^{2M}$ as

$$\Delta_n := \delta_{ij}, \text{ for } n \in [1\ ,\ 2M]_{\mathbb{Z}} \text{ s.t. } \mathcal{A}_n = (i,j).$$

Using these definitions, we wish to obtain the set of optimizers of $\mathbf{P}_2$ in (9) and subsequently define the socially optimal gradient descent dynamics.

$$\begin{aligned} &\mathbf{P}_2 : \\ &\min_{\Delta,z} \sum_{i\in\mathcal{V}} p(a_i, z_i) = \min_{\Delta,z} V(z) \\ &\text{s.t. } z_i = x_i + \sum_{j\in\mathcal{N}^i} (\delta_{ji} - \delta_{ij}),\ \forall\, i \in \mathcal{V}, \\ &\qquad x_i - \sum_{j\in\mathcal{N}^i} \delta_{ij} \geq 0,\ \forall\, i \in \mathcal{V}, \quad \delta_{ij} \geq 0,\ \forall\,(i,j)\in\mathcal{A}. \end{aligned} \tag{9}$$

Observe that $z = x + A\,\Delta$. Note that although $V(z)$ is strictly convex in z, the cost function in $\mathbf{P}_2$ is only convex in (Δ, z). Thus, in general, there may be more than one optimizer for $\mathbf{P}_2$. However, for all the optimizers, the resultant node fractions z_i, for each node $i \in \mathcal{V}$ is unique. The following lemma justifies this claim. We skip its proof due to space limitations.

Lemma 8. (Uniqueness of a Subset of Optimizer Variables in a Class of Convex Optimization Problems). Let $w \in \mathbb{R}^{n_w}$, $y \in \mathbb{R}^{n_y}$ and $q := (w, y)$. Consider the optimization problem

$$\min_{q\in\mathcal{Q}} \bar{f}(q) := \min_{(w,y)\in\mathcal{Q}} f(w).$$

Suppose that the set $\mathcal{Q} \subset \mathbb{R}^{n_w+n_y}$ is convex and the function $f(w)$ is a strictly convex function of w. Then, every optimizer $q^i := (w^i, y^i)$ has the property that $w^i = w^*$, a unique constant. □

Direct application of Lemma 8 to problem $\mathbf{P}_2$ gives us the following result, whose proof we skip here.

Lemma 9. (The Resultant Node Fractions in any Optimal Solution of $\mathbf{P}_2$ are Unique). Consider the problem $\mathbf{P}_2$ in (9) with $x \in \mathcal{S}^N$. Let $\mathcal{Y}$ be the set of all optimizers of $\mathbf{P}_2$. Then $\forall\,(\Delta^*, z^*) \in \mathcal{Y}$, $z^* = x + A\,\Delta^*$ is unique. □

Problem $\mathbf{P}_2$ gives the centralized or socially optimal, network restricted best gradient response given complete knowledge of the population distribution x. Note that $\mathbf{P}_2$ is feasible for each $x \in \mathcal{S}^N$ as $\Delta = 0$ and $z = 0$ is a feasible solution. In Lemma 9, we have established that $z^*(x) = x + A\,\Delta^*$ is uniquely defined for each $x \in \mathcal{S}^N$. Then, we can let the evolution of x as a whole be as

$$D_2(s)\ :\ x(t+1) = x(t) + s\,A\ \Delta^*(a, x(t)) = z^*(x(t)), \tag{10}$$

with $s \in (0, 1]$ the step size parameter. We can also define a continuous-time dynamics as

$$\dot{x} = A\ \Delta^*(a, x) = z^*(a, x) - x, \tag{11}$$

where $z^*(x)$ is the map that gives the unique value of z^* for all optimizers (Δ^*, z^*) of $\mathbf{P}_2$. We refer to the dynamics (10) and (11) as network restricted potential minimization (NRPM). Using ideas similar to those used in proving Lemma 5, one can existence and uniqueness for NRPM, as we state in the following lemma.

Lemma 10. (Existence and Uniqueness of Solutions for Continuous-Time NRPM). Let $a \in \mathbb{R}^N$ be fixed. The state equation in (11) with an initial condition $x(0) \in \mathcal{S}^N$ has a unique solution $\forall\, t \geq 0$. □

We now demonstrate that the dynamics (11) converges to an equilibrium point asymptotically. The proof relies on LaSalle's invariance theorem and we skip it due to space limitations.

Theorem 11. (Continuous-Time NRPM Converges Asymptotically to an Equilibrium Point). For a fixed $a \in \mathbb{R}^N$ and an initial condition $x(0) \in \mathcal{S}^N$, let the evolution of x be governed by (11). Then $\lim_{t\to\infty} x(t) = \overline{x}$, where $\overline{x} \in \mathcal{X}(a)$, with $\mathcal{X}(a)$ defined in (7). □

We can also show that the trajectories of the discrete-time NRPM (10) asymptotically converge to a point in the set of equilibrium points $\mathcal{X}(a)$.

Theorem 12. (Discrete-Time NRPM Converges Asymptotically to an Equilibrium Point). For a fixed $a \in \mathbb{R}^N$, let the evolution of x be governed by $D_2(s)$ in (10) with an initial condition $x(0) \in \mathcal{S}^N$ and let $s \in (0, 1]$. Then $\lim_{t\to\infty} x(t) = \overline{x}$, where $\overline{x} \in \mathcal{X}(a)$, defined in (7). □

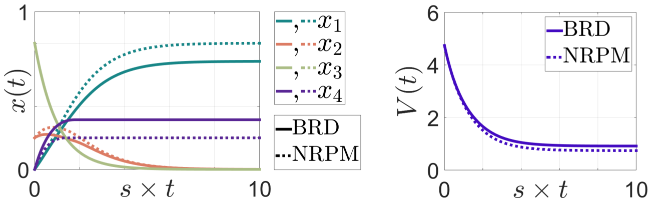
5. DISCUSSION

5.1 A Difference in the Converging Distribution

For a fixed a, we consider the evolution of x to be governed by $D_1(s_1)$ in (3) with $s_1 \to 0$ and $D_2(s_2)$ in (10) with $s_2 \in (0, 1]$ separately for the same initial condition. Then the converging states may not be the same under the two dynamics.

Example 13. Consider a graph with $|\mathcal{V}| = 4$ and $|\mathcal{E}| = 3$. Let the graph be such that node 1 is connected to node 2; node 2 is connected to nodes 1 and 3; node 3 is connected to nodes 2 and 4 and finally node 4 is connected to node

3. Let the initial conditions be given by $x(0) = y(0) = [0, 0.2, 0.8, 0]^T$. Let $a = [0, 2, 5, 2]^T$.



(a) Population Fraction (b) Potential Energy

Fig. 2. Difference in BRD and NRPM.

It is easy to see that no matter how small s is chosen to be, two dynamics provides a different set of optimizers in each time step. Once any fraction of the population has moved to node 4 from node 3, it remains there. This leads to them settling at different equilibria. Simulation results in Figure 2 support this observation. The dynamics were simulated with the said parameters over a period of 10^5 time steps with a step size $s = 10^{-4}$. •

This motivates the discussion about how the dynamics differ fundamentally. In BRD, the infinitesimal agents make their choices with only local knowledge about the system. As they assume that the fraction of the population in the neighboring nodes remains constant, their decision is restricted and hence may not lead to a distribution where the overall potential energy of the system can be reduced further. On the other hand, in NRPM, the infinitesimal agents evolve in a cooperative manner with the reduction of the overall potential in mind. Thus as they evolve, NRPM allows the population as a whole to move to a distribution where the overall potential may be reduced below what is allowed by BRD. This is akin to the concept of "price of anarchy" in game theory.

6. CONCLUSION

We proposed two dynamics that govern the evolution of a population on a network of choices and dealt with their convergence. In the simultaneous BRD, we proposed discrete-time updation rule where the fraction of population in a node takes a decision to redistribute itself amongst its neighboring nodes, assuming that the fraction of population in the neighboring nodes remains constant, in order to minimize a potential function. We also developed a continuous-time differential equation version of the dynamics and showed existence, uniqueness and convergence for an arbitrary initial condition in the simplex. We also proposed a socially optimal dynamics given by the gradient descent of the potential energy. We proposed a discrete-time and a continuous-time version of the same and showed convergence for both.

Future work includes the problem with nodes having limited capacity for holding a population fraction, connections with opinion dynamics and applications to specific domains. Given the convergence phenomenon of both dynamics, we would like to extend this framework to allow for changes in the node parameter at a rate much slower than the rate at which the population distribution converges. We would also like to utilize these results to come up with optimal strategies to change the node parameter to control the distribution of the population.

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