Event-triggered Stabilization for Nonlinear Systems with Center Manifolds

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Abstract— **In this work, we consider event-based implementation of control laws designed for local stabilization of nonlinear systems with center-manifolds. The systems being considered possess linearised models with uncontrollable modes on the imaginary axis. The controller chosen decides both the structure of the center-manifold and its stability. Although the control of systems with center-manifolds is well studied, event-based control of such systems is yet to be probed. This involves the exploration of input-to-state stability (ISS) properties of such systems, with respect to measurement errors. Considering the most general structure for the controller, we prove that a controller that locally asymptotically stabilizes the dynamics on the center-manifold, also renders the overall system locally input-to-state stable (LISS) and find the comparison functions involved in the Lyapunov characterization of ISS. This general characterization required a nonlinear change of variables, involving the center-manifold, which can only be approximately determined in most cases. Because of this, it is found to be unsuitable for designing event-triggered controllers. We then explore an approach that does not resort to this change of variables and present our findings. We discuss the possibility of a simpler relative thresholding mechanism and present simulation results for an illustrative example.**

Index Terms— **Nonlinear systems, Lyapunov stability, Event-based control, Input-to-state stability, Center manifold theory.**

I. INTRODUCTION

In the feedback control of dynamical systems, resources such as computation, sampling and wireless communication are employed. Given a control task and performance requirements, achieving this task using minimum amount of resources is of practical significance and is especially important in large-scale multi-agent and networked control systems. The attention being paid by the control systems community towards implementation aspects of control is evident from the various formulations that have emerged over the years. For instance, minimum attention control, event-triggered control, maximum hands-off control etc. In minimum attention control [1], the number of switchings in control are minimized. In event-triggered control and communication [2], control is updated or the states are broadcasted only when certain events occur in a system. In maximum hands-off control [3], control signals are obtained that have the shortest non-zero support per unit time. Although the formulations and the objectives may seem different, the larger motivation of judicious use of resources remains the same.

The recent renewed interest in the area of event-triggered control is due to the work that appeared in [2]. Since then, the community has explored plants that are linear and nonlinear, both in discrete and continuous time, in deterministic and stochastic and centralized and distributed settings [4], [5]. It has been demonstrated through implementations that, on an average the resources are better utilized in event-triggered implementations, than in time-triggered implementations [6]. Although such wide variety of settings have been explored, the case of event-based control of systems with center manifolds has not been looked at so far. Results relating to linear systems cannot be used directly, as there is no provision to accommodate the dynamics of the center manifold in the triggering conditions. This case came into the authors' notice in the study of the Mobile Inverted Pendulum (MIP) robot [7], whose linearised model has an uncontrollable mode on the imaginary axis, necessitating the center-manifold analysis.

The work in [8] first addressed the control of nonlinear systems, whose linear models possess uncontrollable modes on the imaginary axis. Three particular structures were considered and sufficient conditions for stabilizability of the center-manifold dynamics were derived. A linear controller that stabilizes the controllable subsystem need not stabilize the dynamics on the center-manifold. Along with this linear controller, for a single-input system, a pseudo-control was used in the work of [9], [10] and is chosen to stabilize the dynamics on the center-manifold.

Systems which are input-to-state stable lend themselves easily to event-based implementations. This motivated us to begin our work by investigating input-to-state stability of systems with center-manifolds. Although the control of systems with center-manifolds is well-studied, to the best of our knowledge, the study of such systems in the presence of disturbances and measurement errors has not been investigated so far. In event-triggered control, the error between the last measured state and the current state appears both in the controlled subsystem and the center-manifold dynamics. Although theoretical results for general nonlinear systems suggest that locally asymptotically stable systems are also locally input-to-state stable [11], a constructive Lyapunov characterization and its consequences with respect to event-triggered control, for nonlinear systems with center manifolds, are missing in literature.

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The contributions of this work are the following. Motivated by applications involving the control of nonlinear systems with center-manifolds, we study the general problem of event-triggered implementation of control laws for such systems. We begin by constructively proving that a controller that locally asymptotically stabilizes the dynamics on the center-manifold also renders the overall system locally input-to-state stable with respect to measurement errors. Because of a nonlinear change of variables (which can only be approximately determined in most cases) involved in this general characterization, the derived triggering rules cannot be accurately checked. Proceeding without this change of variables, we design checkable triggering rules but these only ensure, in general, local input-to-state stability of a neighbourhood of the origin. We also show that sufficient conditions in literature ([2], [5]), that rule out Zeno behaviour, are not satisfied by the systems under consideration. We then explore the possibility of a simpler relative triggering rule and present an illustrative example.

II. NOTATIONS AND PRELIMINARIES

We denote by $\mathbb{R}$, the set of real numbers and by $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers. Given two vectors y and w, we denote by $(y; w)$ the concatenation of the two vectors $[y^\top \ w^\top]^\top$. $||\cdot||$ denotes the norm of a vector or the induced norm of a matrix, depending on the argument. Given $A \in \mathbb{R}^{n\times n}$, $A \succ 0$ denotes that A is a positive definite matrix. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to be a class-$\mathcal{K}$ function if $\alpha(0) = 0$ and it is strictly increasing.

Definition 2.1 (Subgradient [12]): A vector $\zeta \in \mathbb{R}^n$ is a subgradient of a function $f : \mathbb{R}^n \to \mathbb{R}$, at a point x in the domain of f, if for all vectors z in the domain of f,

$$f(z) \geq f(x) + \zeta^\top (z - x).$$

The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$.

Lemma 2.2 ([12, page 181]): Let P be a symmetric positive definite matrix and $P = M^\top DM$ be its eigen-decomposition. The subdifferential of the function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $f = \sqrt{x^\top Px}$ at $x = 0$ is $\partial f(0) = \{g \in \mathbb{R}^n : ||g^\top MD^{-\frac{1}{2}}|| \leq 1\}$.

Definition 2.3 (Local input-to-state stability [13]): The system

$$\dot{x} = f(x, d), \ x \in \mathbb{R}^n \text{ and } d \in \mathbb{R}^m$$

with f being locally Lipschitz and $f(0, 0) = 0$, is said to be locally input-to-state stable in the domain $D_x \subset \mathbb{R}^n$ with respect to input d in the domain $D_d \subset \mathbb{R}^m$, if there exists a Lipschitz continuous function $V : D_x \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ and β such that

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$$

and

$$\zeta^\top f(x, d) \leq -\alpha_3(||x||) + \beta(||d||)$$

hold for all $x \in D_x$, $d \in D_d$ and $\zeta \in \partial V(x)$. The function V satisfying the above conditions is called an LISS Lyapunov function.

Compared to input-to-state stability in its global form, the local property is far more relevant and useful in the local analysis of nonlinear systems.

III. PROBLEM FORMULATION

Consider the nonlinear dynamical system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m \tag{1}$$

with $f(0, 0) = 0$. The Taylor series expansion about $x = 0$ and $u = 0$ yields

$$\dot{x} = Ax + Bu + \tilde{f}(x, u) \tag{2}$$

with $\tilde{f}(x, u)$ having the following properties

$$\tilde{f}(0, 0) = 0, \quad \frac{\partial \tilde{f}}{\partial x}(0, 0) = 0 \text{ and } \frac{\partial \tilde{f}}{\partial u}(0, 0) = 0. \tag{3}$$

We are interested in the study of systems (2), which through a linear transformation $x = T(y; z)$ are transformed into the following form

$$\begin{aligned} \dot{y} &= A_1 y + \tilde{g}_1(y, z, u) \\ \dot{z} &= A_2 z + B_2 u + \tilde{g}_2(y, z, u) \end{aligned} \tag{4}$$

where $A_1 \in \mathbb{R}^{k\times k}$, $A_2 \in \mathbb{R}^{(n-k)\times(n-k)}$, the real parts of the eigenvalues of A_1 are zero and the pair (A_2, B_2) is controllable. The control u cannot influence the eigenvalues of the matrix A_1, even though the eigenvalues of A_2 can be arbitrarily placed.

It is easy to see that $\tilde{g}_1(y, z, u)$ and $\tilde{g}_2(y, z, u)$ satisfy

$$\begin{aligned} &\tilde{g}_i(0, 0, 0) = 0, \quad \frac{\partial \tilde{g}_i}{\partial y}(0, 0, 0) = 0, \\ &\frac{\partial \tilde{g}_i}{\partial z}(0, 0, 0) = 0, \quad \frac{\partial \tilde{g}_i}{\partial u}(0, 0, 0) = 0 \text{ for } i = 1, 2. \end{aligned} \tag{5}$$

A. Choice of a controller

Given a set of desired pole locations for the z-subsystem, for a single-input system, there is a unique linear state-feedback controller that achieves this pole-placement. For a multi-input system, it is well known that such a state-feedback controller is not unique. Along with eigenvalues, some desired eigenvectors can also be assigned. Although the origin of the z-subsystem (4) is locally asymptotically stabilized using $u = Kz$, the stability of the origin of the overall system cannot be inferred from the indirect method of Lyapunov, through linearisation. The center-manifold and the dynamics on the center-manifold determine the stability of the overall system and these are determined by the controller chosen. When the system cannot be stabilized by just $u = Kz$, feedback of the form $u = K_{11}z + K_{12}y$ must be employed. The matrix K_{12} is chosen such that the dynamics on the center-manifold is stabilized. If this is inadequate, a pseudo-control $u = K(y; z) = K_{11}z + K_{12}y + \kappa(y)$, $\kappa : \mathbb{R}^k \to \mathbb{R}^m$ can be introduced as in [9], [14]. In the rest of the paper, we use this controller structure, as this is the most general form used in the stabilization of systems with center-manifolds. Denoting $A_2 + B_2K_{11}$ by A_K, we arrive at

$$\begin{aligned}\dot{y} &= A_1y + \tilde{g}_1(y, z, K(y; z)) \\ \dot{z} &= A_Kz + B_2K_{12}y + B_2\kappa(y) + \tilde{g}_2(y, z, K(y; z)).\end{aligned} \tag{6}$$

B. Center Manifold Theory

At this juncture, we recall some important definitions and results from center manifold theory.

Definition 3.1 (Invariant Manifold [15]): For the closed-loop system (6), a k-dimensional manifold $\phi(y, z) = 0$, $\phi : \mathbb{R}^n \to \mathbb{R}^{(n-k)}$ being sufficiently smooth, is said to be locally invariant if $\phi(y(t), z(t)) = 0, \forall\, t \in [0, t_f)$, for some $t_f > 0$, whenever $\phi(y(0), z(0)) = 0$.

For the next few results from center-manifold theory to hold, the cross-coupling linear term $B_2K_{12}y$, between the y and z subsystems must be eliminated. This is done using the change of variables $v = z-Ey$, $E \in \mathbb{R}^{(n-k)\times k}$. As the sum of any eigenvalue of A_1 and any eigenvalue of A_K is non-zero, we use from [10], the result $A_KE-EA_1+B_2K_{12} = 0$ and the notations

$$\begin{aligned}g_1(y, v + Ey, K(y; v{+}Ey)) \\ = \tilde{g}_1(y, v + Ey, K(y; v + Ey))\end{aligned}$$

and $g_2(y, v + Ey, K(y; v + Ey)) =$

$$\begin{aligned}B_2\kappa(y) + \tilde{g}_2(y, v + Ey, K(y; v + Ey)) \\ - E\tilde{g}_1(y, v + Ey, K(y; v + Ey))\end{aligned}$$

to arrive at

$$\begin{aligned}\dot{y} &= A_1y + g_1(y, v + Ey, K(y; v + Ey)) \\ \dot{v} &= A_Kv + g_2(y, v + Ey, K(y; v + Ey)).\end{aligned} \tag{7}$$

Again, it can easily be checked that g_1 and g_2 satisfy conditions (5).

Definition 3.2 (Center Manifold [15]): For the dynamical system (7), a manifold $v = \eta(y)$ is called a center manifold, if

$$\eta(0) = 0, \;\; \frac{\partial \eta}{\partial y}(0) = 0$$

and $v(0) = \eta(y(0))$ implies $v(t) = \eta(y(t)), \;\forall\, t \in [0, t_f)$.

The center-manifold is an invariant manifold for system (7).

Theorem 3.3 (Exitence of a center manifold [15]): If $g_1(y, v)$ and $g_2(y, v)$ are twice continuously differentiable and satisfy conditions in (5), with all eigenvalues of A_1 having zero real parts and all eigenvalues of A_K having negative real parts, then there exists a constant $\delta > 0$ and a continuously differentiable function $h(y)$, defined for all $||y|| \le \delta$, such that $v = h(y)$ is a center manifold for system (7).

System (7) satisfies the hypothesis of Theorem 3.3 for the existence of a k-dimensional center-manifold $v = h(y)$ and the corresponding dynamics on the center manifold is

$$\dot{y} = A_1y + g_1(y, h(y) + Ey, K(y; h(y) + Ey)). \tag{8}$$

We next recall a theorem, known popularly as the *Reduction Theorem*.

Theorem 3.4 (Reduction Theorem [15]): Under the assumptions of Theorem 3.3, if the origin $y = 0$ of the reduced system (8), is locally asymptotically stable (unstable), then the origin of the full system (7) is locally asymptotically stable (unstable).

In determining the stability on the center-manifold, $h(y)$ is found upto an approximation by solving the partial differential equation

$$\begin{aligned}0 = A_Kh(y) + g_2(y, h(y) + Ey, K(y; h(y) + Ey)) - \\ \frac{\partial h(y)}{\partial y}(A_1y + g_1(y, h(y) + Ey, K(y; h(y) + Ey))).\end{aligned} \tag{9}$$

C. Event-based control and measurement errors

Although results exist in literature that design controllers that not only stabilize the dynamics on the center manifold but also obtain a particular structure for the center-manifold dynamics, event-based control of such systems is still an unexplored area. In event-based control, between two events, in the interval $[t_k, t_{k+1})$, the control is held constant to $u = K(y(t_k); v(t_k) + Ey(t_k))$, t_k being the k^{th} triggering instant. With the measurement error $e_y = y(t_k) - y(t)$ and $e_v = v(t_k) - v(t)$, the control can be written as $u = K(y+e_y; v+e_v+E(y+e_y))$. The closed-loop system with $K_1 = [K_{11} \;\; (K_{11}E + K_{12})]$ and $e = (e_y; e_v)$ is

$$\begin{aligned}\dot{y} &= A_1y + g_1(y, v + Ey, u) \\ \dot{v} &= A_Kv + B_2K_1e + g_2(y, v + Ey, u).\end{aligned} \tag{10}$$

For further analysis, we introduce new coordinates $w = v - h(y)$ (the need for which is explained in Section V). Using the notation $e_w = w(t_k) - w(t)$, $e_h = h(y(t_k)) - h(y(t))$, $e = (e_y; e_w + e_h)$ and the control $u = K(y+e_y, w+h(y)+ e_w + e_h + E(y + e_y))$, the dynamics gets transformed to

$$\begin{aligned}\dot{y} &= A_1y + g_1(y, w + h(y) + Ey, u) \\ \dot{w} &= A_K(w + h(y)) + B_2K_1e + g_2(y, w + h(y) + Ey, u) \\ &\quad - \frac{\partial h(y)}{\partial y}(A_1y + g_1(y, w + h(y) + Ey, u)).\end{aligned} \tag{11}$$

Subtracting (9) from (11) and using the following notations

$$\begin{aligned}N_1(y, w, e) &= g_1(y, w + h(y) + Ey, K(y + e_y; w + h(y) \\ &\quad + e_w + e_h + E(y + e_y))) - g_1(y, h(y) + Ey, \\ &\quad K(y; h(y) + Ey)) \\ N_2(y, w, e) &= g_2(y, w + h(y) + Ey, K(y + e_y; w + h(y) \\ &\quad + e_w + e_h + E(y + e_y))) - g_2(y, h(y) + Ey, \\ &\quad K(y; h(y) + Ey)) - \frac{\partial h(y)}{\partial y}(A_1y + g_1(y, w \\ &\quad + h(y), u) - (A_1y + g_1(y, h(y) + Ey, \\ &\quad K(y; h(y) + Ey))))\end{aligned}$$

we obtain

$$\begin{aligned}\dot{y} &= A_1y + g_1(y, h(y) + Ey, K(y; h(y) + Ey)) \\ &\quad + N_1(y, w, e) \\ \dot{w} &= A_Kw + B_2K_1e + N_2(y, w, e).\end{aligned} \tag{12}$$

It is easy to deduce that for $i = 1, 2$

$$N_i(y, 0, 0) = 0, \frac{\partial N_i}{\partial w}(0, 0, 0) = 0 \text{ and } \frac{\partial N_i}{\partial e}(0, 0, 0) = 0. \tag{13}$$

With N_i satisfying these conditions, it can be shown that there exist $\delta_{yw} > 0$ and $\delta_e > 0$ such that, whenever

$$||(y; w)|| \leq \delta_{yw} \text{ and } ||e|| \leq \delta_e \tag{14}$$

we have for $i = 1, 2$,

$$||N_i|| \leq k_i||(w; e)|| \leq k_i(||w|| + ||e||).$$

The constants k_i can be made arbitrarily small by decreasing δ_{yw} and δ_e. Note that the stability properties of system (12) are the same as that of the system (1) because of the smooth change of coordinates relating the two systems.

For the development of event-triggered implementation, we first choose a controller (as discussed in subsection III-A) that locally asymptotically stabilizes the overall system. It is known that a controller that locally asymptotically stabilizes the closed-loop system also renders the overall system LISS [11]. However, for designing event-triggered control, the class-$\mathcal{K}$ functions α_3 and β (Definition 2.3) have to be determined and the following section presents these functions for systems with center-manifolds.

IV. LOCAL INPUT-TO-STATE STABILITY OF SYSTEMS WITH CENTER-MANIFOLDS

In this section, we show that a controller that locally asymptotically stabilizes the dynamics on the center-manifold also yields LISS of the overall system. We also find the class-$\mathcal{K}$ function α_3 and β involved in the LISS characterization (Definition 2.3).

Proposition 4.1: Under the assumption that the functions g_1 and g_2 of system (7) satisfy conditions (5), if the origin $y = 0$ of the center-manifold dynamics (8) is locally asymptotically stable, then the overall system (12) is locally input-to-state stable with respect to the error e.

Proof: By the converse Lyapunov theorem, local asymptotic stability of the equilibrium point of the center manifold dynamics (8) implies the existence of a continuously differentiable Lyapunov function $V_1 : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}$ functions α_4, α_5 such that

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V_1}{\partial y}(A_1 y + g_1(y, h(y) + Ey, K(y; h(y) + Ey))) \\ &\leq -\alpha_4(||y||) \quad \text{and} \\ &\qquad \left|\left|\frac{\partial V_1}{\partial y}\right|\right| \leq \alpha_5(||y||) \leq k\end{aligned}$$

for some $k > 0$, in a neighbourhood of the origin. Also, since the matrix A_K is Hurwitz, for every $Q \succ 0$ there exists a $P \succ 0$ such that $A_K^\top P + PA_K = -Q$. Consider the LISS Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

$$V(y, w) = V_1(y) + \sqrt{w^\top Pw}. \tag{15}$$

The function V is differentiable everywhere except on the set $N_d = \{(y, w) \in \mathbb{R}^n \ : \ w = 0\}$. On the set $\mathbb{R}^n \setminus N_d$, the subdifferential is the derivative of V. Taking the derivative of V along the trajectories of system (12) on the set $\mathbb{R}^n \setminus N_d$

$$\begin{aligned}\dot{V} &= \frac{\partial V_1}{\partial y}\dot{y} + \frac{1}{2\sqrt{w^\top Pw}}\left(\dot{w}^\top Pw + w^\top P\dot{w}\right) \\ &= \frac{\partial V_1}{\partial y}(A_1 y + g_1(y, h(y) + Ey, K(y; h(y) + Ey) \\ &\quad + N_1(y, w, e)) + \frac{1}{2\sqrt{w^\top Pw}}((A_K w + B_2 K_1 e + N_2(y, \\ &\quad w, e))^\top Pw + w^\top P(A_K w + B_2 K_1 e + N_2(y, w, e))) \\ &\leq -\alpha_4(||y||) + kk_1(||e|| + ||w||) - \frac{w^\top Qw}{2\sqrt{w^\top Pw}} \\ &\qquad + \frac{1}{\sqrt{w^\top Pw}}(w^\top PB_2 K_1 e + w^\top PN_2(y, w, e)) \\ &\leq -\alpha_4(||y||) - \frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| + kk_1(||e|| + ||w||) \\ &\qquad + \frac{||PB_2K_1||}{\sqrt{\lambda_{min}(P)}}||e|| + \frac{k_2\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}(||e|| + ||w||).\end{aligned}$$

With $s_f \in (0, 1)$, we obtain

$$\begin{aligned}\dot{V} &\leq -\alpha_4(||y||) - (1 - s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| \\ &\quad + \left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} - s_f\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}\right)||w|| \\ &\quad + \left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} + \frac{||PB_2K_1||}{\sqrt{\lambda_{min}(P)}}\right)||e||\end{aligned}$$

The constants k_1 and k_2 can be made arbitrarily small by choosing δ_{yw} and δ_e in equation (14), such that

$$\left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} - s_f\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}\right) \leq 0.$$

Therefore

$$\begin{aligned}\dot{V} &\leq -\alpha_4(||y||) - (1 - s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w|| \\ &\quad + \left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} + \frac{||PB_2K_1||}{\sqrt{\lambda_{min}(P)}}\right)||e||.\end{aligned}$$

Notice that the function $\alpha_D(||(y; w)||) = \alpha_4(||y||) + (1 - s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w||$ is a class-$\mathcal{K}$ function of $||(y; w)||$ and the function $\beta_G(||e||) = \left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} + \frac{||PB_2K_1||}{\sqrt{\lambda_{min}(P)}}\right)||e||$ is a class-$\mathcal{K}$ function of $||e||$. Using Lemma 2.2, $\partial V(0) = \{(0; g) \in \mathbb{R}^n : g \in \mathbb{R}^{n-k} \text{ and } ||D^{-\frac{1}{2}}Mg|| \leq 1\}$. The inner product $\zeta^\top(\dot{y}; \dot{w})$ evaluated at the origin is

$$\begin{aligned}&\zeta^\top(0; BK_1e + N_2(0, 0, e)) \\ &\quad = \zeta^\top M^\top D^{-\frac{1}{2}} D^{\frac{1}{2}} M(0; BK_1e + N_2(0, 0, e)) \\ &\quad \leq ||D^{-\frac{1}{2}} MP(BK_1e + N_2(0, 0, e))|| \\ &\quad \leq \left(\frac{||PBK_1||}{\sqrt{\lambda_{min}(P)}} + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}\right)||e|| \leq \beta_G(||e||)\end{aligned}$$

holds for all $\zeta \in \partial V(0)$. From Definition 2.3, V is a local ISS Lyapunov function for system (12). Therefore the origin of system (12) is locally input-to-stable with respect to the error e, when $||(y;w)|| \leq \delta_{yw}$ and $||e|| \leq \delta_e$. ∎

This proposition generalizes the *Reduction Theorem* (Theorem 3.4), as in the absence of disturbance e, local asymptotic stability of the overall system is recovered.

V. EVENT-TRIGGERED CONTROL

In the previous section, we showed LISS of system (12). In this section, we consider the relative triggering mechanism for event-triggered implementation proposed in [2]. The local ISS Lyapunov function yielded

$$\dot{V} \leq -\alpha(||(y;w)||) + \beta(||e||)$$

with $\alpha(||(y;w)||) = -\alpha_4(||y||) - (1-s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||w||$ and $\beta(||e||) = \left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} + \frac{||PB_2K_1||}{\sqrt{\lambda_{min}(P)}}\right)||e||$. Now, using the simple triggering rule $\beta(||e||) \leq \sigma\alpha(||(y;w)||)$, we obtain $\dot{V} \leq -(1-\sigma)\alpha(||(y;w)||)$. For $\sigma \in (0,1)$, asymptotic stability is recovered. Note that this triggering condition is not easy to check as the coordinate $w = z - h(y)$, in most cases, can only be computed upto an approximation, because $h(y)$ is determined by solving (9). This brings into question the utility of using the coordinate transformation $w = z - h(y)$.

The conditions (13) and the change of variables from z to $w = z - h(y)$ are crucial in the proof of Proposition 4.1 and the following discussion brings out their utility. Proceeding without the change of variables from (7), with

$$\begin{aligned} N_1 &= g_1(y, v + Ey, K(y+e_y; v+e_v+E(y+e_y)) \\ &\qquad - g_1(y, h(y)+Ey, K(y; h(y)+Ey)) \\ N_2 &= g_2(y, v+Ey, K(y+e_y; v+e_v+E(y+e_v))) \end{aligned}$$

which satisfy $N_i(0,0,0) = 0$, $\frac{\partial N_i}{\partial y}(0,0,0) = 0$, $\frac{\partial N_i}{\partial v}(0,0,0) = 0$ and $\frac{\partial N_i}{\partial e}(0,0,0) = 0$, we can show the existence of $\delta_{yv} > 0$ and $\delta_e > 0$ such that, whenever $||(y;v)|| \leq \delta_{yv}$ and $||e|| \leq \delta_e$, we have for $i = 1,2$,

$$||N_i|| \leq k_i||(y;v;e)|| \leq k_i(||y|| + ||v|| + ||e||). \quad (16)$$

The constants k_i can be made arbitrarily small by decreasing δ_{yv} and δ_e. With $s_y \in (0,1)$, this would result in

$$\begin{aligned} \dot{V} &\leq -(1-s_y)\alpha_4(||y||) - (1-s_f)\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}||v|| \\ &\quad + \left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} + \frac{||PB_2K_1||}{\sqrt{\lambda_{min}(P)}}\right)||e|| \\ &= -\alpha(||(y;v)||) + \beta(||e||) \end{aligned} \quad (17)$$

whenever

$$\left(\left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}\right)||y|| - s_y\alpha_4(||y||)\right) \leq 0, \quad (18)$$

and k_1, k_2 and λ are chosen such that

$$\left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} - s_f\frac{\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}\right) \leq 0.$$

Note that if $\alpha_4(||y||) \in \mathcal{O}(||y||^p)$, $p \leq 1$, then k_1 and k_2 can again be chosen such that (18) is true and the origin is locally input-to-state stable. If not, then local input-to-state stability of only a set of the form

$$\begin{aligned} S = \{y \in \mathbb{R}^k \mid &\left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}\right)||y|| \\ &- s_y\alpha_4(||y||) \geq 0\} \end{aligned} \quad (19)$$

can be inferred. The function α_4 guaranteed by the converse Lyapunov theorem can be any class-$\mathcal{K}$ function and may or may not belong to $O(||y||^p)$, $p \leq 1$. The change of variables from v to w allows us to incorporate all class-$\mathcal{K}$ functions. The conditions (13) are crucial for this incorporation and without the change of variables, conditions of the form (13) could not have been obtained.

From (17), we can derive an implementable event-triggering scheme

$$\beta(||e||) \leq \sigma\alpha(||(y;v)||). \quad (20)$$

Unlike the triggering condition involving the variable w, this involves y and v, which can be obtained from x (which is actually measured), through a linear transformation.

Next, we present the following corollaries of Proposition 4.1

Corollary 5.1: Under the assumption that the functions g_1 and g_2 of system (7) satisfy conditions (5), if the origin $y = 0$ of the center manifold dynamics (8) is locally asymptotically stable and there exists a Lyapunov function such that $\alpha_4 \in \mathcal{O}(||y||^p)$, $p \leq 1$, then the overall system (10) is locally asymptotically stable under event-triggered implementation $\beta(||e||) \leq \sigma\alpha(||(y;v)||), \sigma \in (0,1)$.

This Corollary presented next is important as in many cases, α_4 turns out to be a polynomial with degree greater than one.

Corollary 5.2: Under the assumption that the functions g_1 and g_2 of system (7) satisfy conditions (5), if the origin $y = 0$ of the center manifold dynamics (8) is locally asymptotically stable and there exists a Lyapunov function such that $\alpha_4 \in \mathcal{O}(||y||^p)$, $p > 1$, then the set S in (19) of the overall system is locally asymptotically stable under event-triggered implementation $\beta(||e||) \leq \sigma\alpha(||(y;v)||), \sigma \in (0,1)$.

Note :- The comparison functions α^{-1} and β meet the sufficient regularity condition of Lipschitzness over compact sets (presented in [2], [5]) that rule out non-existence of Zeno behaviour, only when $\alpha_4 \in \mathcal{O}(||y||^p)$, $p \leq 1$. When $\alpha_4 \in \mathcal{O}(||y||^p)$, $p > 1$, the sufficient conditions are not satisfied and no conclusion can be drawn about the existence or non-existence of Zeno behaviour through them.

VI. EXAMPLE WITH SIMULATION

From equation (17) we can observe that a much simpler triggering condition of the form $||e|| \leq \sigma||v||$ can be used, which guarantees that $\dot{V} < 0$ outside the set $\bar{S} = \{y \in \mathbb{R}^k \mid \left((1+\sigma)\left(kk_1 + \frac{k_2\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}\right)||y|| - s_y\alpha_4(||y||)\right) \geq 0\}$ when k_1 and k_2 can be chosen such that $\left((1+\sigma)\left(kk_1 + k_2\frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}\right) - \frac{s_f\lambda_{min}(Q)}{2\sqrt{\lambda_{max}(P)}}\right) \leq 0.$

Next, we take up an example to implement this triggering rule. Consider the system

$$\begin{aligned}\dot{y} &= 10^5(3y^3 - yu)\\ \begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -2 & 3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u + \begin{bmatrix}y^2\\ 0\end{bmatrix}.\end{aligned}$$

This system is open-loop unstable with poles at $\{1, 2\}$. The linear feedback controller $u = [1 \;\; -4]x$ is used to place the poles at $-0.5 \pm i0.0866$. The closed-loop system is

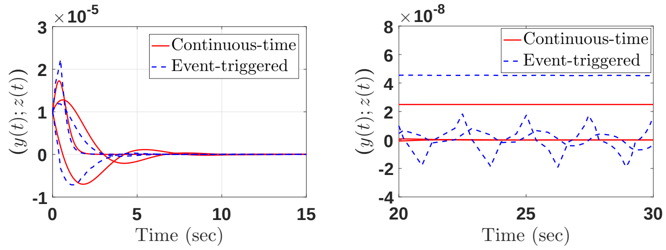
$$\begin{aligned}\dot{y} &= 10^5(3y^3 - y(z_1 - 4z_2))\\ \begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}y^2\\ 0\end{bmatrix}.\end{aligned} \tag{21}$$

The equations of the one-dimensional center-manifold are given by $z_1 = h_1(y) = y^2 + \mathcal{O}(y^4)$ and $z_2 = h_2(y) = -y^2 + \mathcal{O}(y^4)$. The dynamics on the center-manifold is

$$\dot{y} = -2 \times 10^5 y^3 + \mathcal{O}(y^5) \tag{22}$$

The Lyapunov function $V_1(y) = \frac{1}{2}y^2$ can be used to show the local asymptotic stability of (22), as $\dot{V}_1 = y\dot{y} = -2\times 10^5 y^4 + \mathcal{O}(y^6) < 0$, close to the origin. For the z-subsystem of (21) we can choose a Lyapunov function of the form $\sqrt{z^\top P z}$, with $P \succ 0$ obtained by solving the Lyapunov equation for a specific $Q \succ 0$. Using Theorem 3.4 and Proposition 4.1, local asymptotic stability and LISS with respect to measurement errors respectively can be inferred.

Simulation results of continuous time and event-triggered implementation of the control law using the triggering rule $||e|| \leq 0.4||z||$ are presented in Fig. 1. The initial condition for the system was set to $(10^{-5}, 10^{-5}, 10^{-5})$. It can be observed that the trajectories of the system, in event-triggered implementation, converge to a set. The minimum time between two consecutive triggers was found to be 0.1606 seconds. Although the triggering instants do not accumulate for this initial condition, non-existence of Zeno-behaviour has to be theoretically shown, for the triggering rule to be deemed implementable.



(a) (b)

Fig. 1: Evolution of the states of the system in Example (1) under under continuous time implementation and event-triggered implementation of the control law.

VII. CONCLUSIONS

In this work, event-based implementation of control laws designed for local stabilization of nonlinear systems with center-manifolds was considered. A constructive proof showing that a controller that locally asymptotically stabilizes a system also yields local ISS is presented. Because a change-of variables involving the center-manifold was employed, the Lyapunov characterization of LISS obtained was not found to be useful for event-triggered control. This prompted the search for other ways of obtaining LISS characterizations, which yielded checkable triggering conditions. However, LISS of only a neighbourhood of the origin could be concluded. It was shown that comparison functions in the LISS characterisations do not meet the sufficient regularity conditions of Lipschitzness on compact sets already established in literature, which rule out Zeno behaviour. An alternative simpler relative thresholding mechanism was explored and employed for event-based control of an illustrative example. Designing event-triggered controllers that recover asymptotic stability of the origin and showing the non-existence of Zeno behaviour form part of our future work.

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