Opinion Dynamics for Utility Maximizing Agents with Heterogeneous Resources

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***Abstract*— In this paper we propose a continuous-time non-linear model of opinion dynamics. One of the main novelties of our model is that it costs resources for an agent to express an opinion. Each agent receives a utility based on the complete opinion profile of all agents. Each agent seeks to maximize its own utility function by suitably revising its opinion and the proposed dynamics arises from all agents simultaneously doing this. For the proposed model, we show ultimate boundedness of opinions. We also show stability of equilibrium points and convergence to an equilibrium point when all agents are non-contrarian. We give conditions for the existence of a consensus equilibrium and analyze the role that resources play in determining the social power of the agents in terms of the deviation of the consensus value from the agents' internal preference. We also carry out a Nash equilibrium analysis of the underlying game and show that when all agents are non-contrarian, the set of equilibria of the opinion dynamics is the same as the set of Nash equilibria for the underlying game. We illustrate our results using simulations.**

***Index Terms*— Opinion dynamics, Multi-agent systems, Utility maximization.**

I. Introduction

Opinion dynamics in social groups or networks is an important research topic with applications in diverse areas such as sociology, economics, public health, transportation and other engineering disciplines. For this reason, a diverse set of communities has studied opinion dynamics for several decades. The controls community has also made significant contributions to this field in recent years.

Literature Review: Some of the first models of opinion dynamics for finitely many agents are the French-DeGroot model [1], [2] and the Abelson model [3], which are essentially consensus dynamics. Taylor's model [4] and the Friedkin-Johnsen model [5] extend the consensus dynamics models to include stubborn or prejudiced agents to explain the opinion cleavage phenomenon. In all these models, the topology of the social network remains constant. In contrast, in the Hegselmann-Krause(HK) model [6], at each time instant, each agent is influenced by only those agents whose opinions are within a confidence bound of its own opinion. The Altafini model [7] captures antagonistic relationships among the agents by considering signed graphs. Gossip-based models consider the case of asynchronous interactions among the agents. The Deffuant-Weisbuch (DW) model [8] is a gossip-based counterpart of the HK model, where not all agents exchange their opinions simultaneously, and only a random choice of agents interact at each time step. These are some of the fundamental models which serve as a base for further extensions in the literature. Readers can find a detailed summary of the models mentioned above and other recent contributions in [9]–[11].

The concept of *social power* was first introduced for the French-DeGroot model. The DeGroot-Friedkin (DF) model, first proposed in [12], captures a combination of two processes: the process of opinion formation and the process of social power evolution. A generalized DF model has been proposed in [13] and a modified version of the DF model, considering stubborn agents, was proposed in [14].

A game-theoretic or utility maximization approach for modeling opinion dynamics is still in a nascent stage, with only a few works appearing so far. An opinion dynamics model with stubborn agents in which each agent minimizes a cost function using the *best response dynamics* was proposed in [15]. A recent work [16] considers a dynamic influence maximization game in which multiple competing parties called *players* try to allocate their fixed resources over a set of *individuals* (who hold opinions about each *player*) to maximize their utility in the long term. A game theoretical analysis of the asynchronous HK model was carried out in [17]. A model which captures the effect of co-evolution of opinions formed and actions taken by the agents was proposed in [18]. Here the agents aim to coordinate their actions (depending on their own opinion) with the actions of others using best response dynamics, and agents try to maximize their payoffs. A continuous time non-linear opinion dynamics model, proposed in [19], was used for tuning the mutual cooperative behavior of the agents in a repeated game, where the agents rely on rationality and reciprocity to take strategic decisions. A discrete-time opinion dynamics model, with a game theoretic structure, in which every agent incurs a cost for forming an opinion (the cost function consists of a conformity cost and a manipulation cost) was proposed in [20]. Here the goal of every agent is to hold an opinion which minimizes this cost. In this model, opinions converge to the unique Nash equilibrium. An *expressed-private-opinion* model was proposed in [21], in which every agent's private opinion is influenced by its neighbors' opinions but under the pressure of conforming with everyone else, the agent expresses an opinion different from its private opinion. The model that we have proposed in this paper also considers expressed opinion and internal preference opinion for an

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agent, but in our case, the preference of an agent is static, and every agent wants to have its opinion close to its preference.

Contributions: Unlike most of the literature on opinion dynamics, we start with utility functions for each agent that depend on the collective opinion profile. The utility function of each agent includes a resource penalty for expressing an opinion of large magnitude. The proposed opinion dynamics is then each agent ascending along the gradient of its utility function while assuming other agents do not revise their opinions. The resulting continuous time opinion dynamics model is non-linear due to the resource penalty. Without the resource penalty term, the proposed dynamics is similar to the *Taylor's* model [4]. Such a penalty is unique to our model and is motivated by the fact that expressing an opinion or influencing others with an opinion requires resources such as wealth, time, social influence or combinations of them.

For the proposed model, we show ultimate boundedness of opinions. For *non-contrarian* agents, we analyze the stability of the equilibrium points and show convergence of opinions to an equilibrium. We provide a necessary and sufficient condition for the existence of consensus equilibrium for the dynamics. In case of convergence to a consensus equilibrium, we also analyze the deviation of the consensus value from each agent's internal preference and relate it to the heterogeneous resources of the agents. We further carry out game-theoretic analysis of our model. For non-contrarian agents, we show that the set of equilibria of the dynamics is the same as the set of Nash equilibria of the underlying game. Compared to the game theoretic formulations of opinion dynamics [15]–[20], our primary contribution is the resource penalty in the utility functions and analysis of the effects of heterogeneity in the resources available to each agent.

Organization of the paper: The rest of the paper is organized as follows, Section II formally introduces the utility function and the opinion dynamics model. Section III contains the analysis of the long-term behavior of the proposed dynamics, such as ultimate boundedness of the opinions, stability of equilibria and convergence of opinions for *non-contrarian* agents. Section IV contains the analysis of the consensus equilibrium in our model. In this section, we also give results connecting the equilibria of the dynamics and the Nash equilibria of the underlying game. In Section V, we demonstrate our results using simulations. Concluding remarks appear in Section VI.

Notation: Throughout the paper, we use non-bold letters for denoting scalars, bold lowercase letters for denoting vectors, and bold uppercase letters for denoting matrices. The sets of real numbers, non-negative real numbers and positive real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$, respectively. Let $\mathbf{1} \in \mathbb{R}^n$ denote a vector with all elements equal to one. For a set $\mathcal{S}$, $\mathcal{S}^n$ denotes the Cartesian product of $\mathcal{S}$ with itself n times. The null set is denoted by $\varnothing$. For a function of time $f(t)$, $D^+ f(t)$ represents the upper-right-hand derivative, i.e., $D^+ f(t) = \limsup_{h \to 0+} \frac{f(t+h)-f(t)}{h}$.

II. Modeling and Problem Statement

Consider a set $\mathcal{A} := \{1, \dots, n\}$ of n agents. We seek to model and study the dynamics of opinions expressed by the agents on a single topic. We start with a utility function for each agent and obtain the proposed opinion dynamics model, assuming that each agent myopically seeks to maximize its utility function. For each agent, the utility function depends on its preferred opinion, the influence of other agents and resource penalty. Thus, the coupling in the utility functions creates the coupling in the opinion dynamics of the agents.

Opinions, Utility Function and its Parameters: We denote the *expressed opinion* of agent i at time t on the topic with $z_i(t) \in \mathbb{R}$. For brevity, we skip the time argument wherever there is no confusion. The vector $\mathbf{z} := [z_1, \cdots, z_n]^\top \in \mathbb{R}^n$ represents the stacked opinions z_i of all agents $i \in \mathcal{A}$. The complete opinion profile $\mathbf{z}$ determines the utility for each agent $i \in \mathcal{A}$. We allow for heterogeneous agents, each having different parameters in their utility function. We first present the utility function of agent i, then describe the various parameters and then provide motivation for the structure of the utility functions. In particular, we let the *utility function* of agent i be

$$U_i(\mathbf{z}) = \frac{-w_i r_i}{2B}(z_i - p_i)^2 - \frac{c_i}{2}\left(\sum_{k \in \mathcal{A}} \frac{r_k}{B}(z_i - z_k)^2\right) - \frac{1}{4r_i} z_i^4. \tag{1}$$

While agent i expresses the opinion z_i, its internal *preference* on the topic is $p_i \in \mathbb{R}$. The scalar $w_i \in \mathbb{R}_{>0}$ represents the importance that agent i attaches to its internal preference on the topic. The scalar $c_i \in \mathbb{R}$, called as the *conformity* weight, quantifies agent i's desire to conform with the others. Positive c_i implies that agent i wants to conform with others, whereas negative c_i implies that the agent is a contrarian. An agent $i \in \mathcal{A}$ is said to be *non-contrarian* if $c_i \geq 0$. In our model, expressing an opinion costs resources. Each agent i has *resources* $r_i \in \mathbb{R}_{>0}$, which can represent quantities such as wealth, time, social influence or combinations of them. The parameter r_i can be more precisely interpreted as the rate at which agent i "uses" the resources or exerts influence with. In this paper, we assume the parameters r_i are constant in time. The total resources available to the collection of agents is $B := \sum_{k \in \mathcal{A}} r_k$. Thus, the factor (r_i/B) is the resources available to agent i relative to the entire social group, and is a measure of agent i's social clout. Throughout the paper, we make the following assumptions about the aforementioned parameters.

***Assumption** 2.1: (**Parameters' Signs**).* For each agent $i \in \mathcal{A}$, we assume $p_i \in \mathbb{R}$, $w_i \in \mathbb{R}_{>0}$, $c_i \in \mathbb{R}$ and $r_i \in \mathbb{R}_{>0}$. •

***Remark** 2.2: (**Motivation for the Utility Function**).* Recall that each agent i myopically seeks to maximize its utility function U_i. Thus, the three terms in (1) could be interpreted as forces on agent i's expressed opinion z_i towards different things. The first term drives the opinion towards its internal preference p_i. The strength of this force is directly proportional to the importance weight w_i as well as the relative social resources (r_i/B) available to agent i. The second term drives the agents towards or against conformity depending on the sign of their conformity weight. The third term (called the *resource penalty*) prevents the agent from holding too extreme opinions. In particular, the greater the resources that agent i has, the more extreme opinions it can hold. Similarly, we see that the greater the relative resources (r_k/B) of agent k, the greater is its influence on the opinions of other agents i.

In this paper, we consider a simple form of the utility function. However, one may consider a more general and complicated class of utility functions too. We choose a quartic term for the resource penalty. A quadratic penalty term would lead to a linear term in the dynamics, which could be absorbed in the remaining terms. Another reason for choosing a quartic penalty function is to ensure that the opinions of the agents are bounded.

Factorizing the coefficient of the first term as $w_i(r_i/B)$ lets us separate out the agent's importance (or bias, stubbornness) towards its internal preference and the agent's ability, due to its resources, to choose an opinion closer to its preference. One could choose $p_i = z_i(0)$ or any other value that the agent prefers. •

Opinion Dynamics: Since each agent $i \in \mathcal{A}$ myopically revises its opinion z_i in response to others' opinions, the opinion vector $\mathbf{z}$ would generally, change with time. We assume that at each time instant agent i revises its opinion by doing a gradient ascent of its utility function U_i, given in (1), with respect to its opinion z_i. Thus, for each $i \in \mathcal{A}$, we have

$$\dot{z}_i = -\frac{w_i r_i}{B}(z_i - p_i) - c_i\left(\sum_{k\in\mathcal{A}} \frac{r_k}{B}(z_i - z_k)\right) - \frac{z_i^3}{r_i}. \quad (2)$$

Note that the first two terms in the opinion dynamics (2) are similar to the *Taylor*'s model [4]. However, this is incidental due to the specific form of the utility function (1). The main difference between our model and the *Taylor's* model is the resource penalty term in our model. Additionally, we are modeling the *persuasibility* constants defined in [4] in terms of the resources r_i and the weights w_i. Notice that a more complex choice of the utility function would result in a different opinion dynamics.

Also notice that in the opinion dynamics (2), there is no social network or graph. However, one can easily incorporate a social network by having an additional factor of $a_{i,k}$ (element of the adjacency matrix of the social network) multiplying the term $(z_i - z_k)^2$ in the utility function (1). However, we choose not to do it in this paper as we seek to focus on the effect of resources available to agents on the dynamics. The model (2) is applicable without any modification to closely knit groups, where the social network is a complete graph. Incidentally, the model (2) is also applicable to very large groups of agents, where each agent is primarily influenced by only publicly known or broadcasted social signals. To see this, we can rewrite (2) equivalently as

$$\dot{z}_i = y_i := S_i(z_i) + C_i(\mathbf{z}), \ \forall i \in \mathcal{A}, \quad (3)$$

with

$$S_i(z_i) := -\frac{w_i r_i}{B}[z_i - p_i] - \frac{z_i^3}{r_i}, \quad (4a)$$

$$C_i(\mathbf{z}) := c_i[\bar{z} - z_i] \quad (4b)$$

and where

$$\bar{z} := \sum_{k\in\mathcal{A}} \frac{r_k}{B} z_k \quad (5)$$

is the weighted average of the opinions of all agents in the social group. So, it suffices for each agent $i \in \mathcal{A}$ to know only $\bar{z}$ and be unaware of other individual agents' opinions. Note that $\forall i \in \mathcal{A}$, the *self function* $S_i(\cdot)$ depends only on the agent's own opinion, its preference p_i and its resources r_i. On the other hand, $\forall i \in \mathcal{A}$, *consensus function* $C_i(\cdot)$ depends only on the deviation of the agent's opinion z_i from $\bar{z}$, the weighted average of opinions of all agents in $\mathcal{A}$. If $c_i > 0$, the agent values conformity and hence $C_i(\cdot)$ drives z_i towards consensus. On the other hand, if $c_i < 0$, the agent is a contrarian and hence $C_i(\cdot)$ drives z_i away from consensus. If $c_i = 0$, $\bar{z}$ does not affect the evolution of z_i.

Now, notice that for each $i \in \mathcal{A}$, $S_i(\cdot)$ is a strictly decreasing function with $\lim_{z_i\to-\infty} S_i(z_i) = \infty$ and $\lim_{z_i\to\infty} S_i(z_i) = -\infty$. Thus, for each $i \in \mathcal{A}$, $S_i(\cdot)$ has exactly one real root. Let us denote the real root of $S_i(\cdot)$ as m_i, *i.e.*, $S_i(m_i) = 0$. Moreover, by considering $S_i(0)$ and $S_i(p_i)$, we can verify that $0 \leq |m_i| \leq |p_i|$ and $m_i p_i \geq 0$. Also note that since r_i's are strictly positive and B is the sum of all r_i's, $\bar{z}$ is essentially a convex combination of all z_i.

Objective: For the opinion dynamics proposed in this paper, our main objective is to capture the effect of resources available to the agents on their opinion evolution. In particular, we first aim to analyze the long-term behavior exhibited by the opinions, more specifically, stability and asymptotic properties of (2). We also aim to investigate the properties of the equilibrium points of (2), including consensus equilibria and Nash equilibria of the underlying game.

III. Long Term Behavior of Opinions

In this section, we analyze the long-term behavior of opinions such as ultimate boundedness and stability of the equilibrium points of the dynamics under different conditions. We also give convergence guarantees for the opinion trajectories starting from an arbitrary initial opinion vector. For ease of discussion, we denote the set of equilibrium points of (2) as

$$\mathcal{E} := \{\mathbf{z} \in \mathbb{R}^n \mid \dot{\mathbf{z}} = \mathbf{0}\}. \quad (6)$$

A. Ultimate Boundedness

If the opinions of all agents evolve according to the dynamics given in (2) then the resource penalty forces the opinion of every agent $i \in \mathcal{A}$ to be bounded for all time $t \geq 0$. In fact, the opinions of all agents are ultimately bounded, which we state formally in the following result.

*Theorem 3.1: (**Ultimate Boundedness of Opinions**).* Consider the dynamics (2). For all initial opinions $\mathbf{z}(0) \in \mathbb{R}^n$, $\exists \eta \geq 0$ and $\exists T(\mathbf{z}(0)) \geq 0$ such that $|z_i(t)| \leq \eta$, $\forall t \geq T(\mathbf{z}(0))$, $\forall i \in \mathcal{A}$.

Proof: Consider the following radially unbounded positive definite Lyapunov like function

$$V := V(\mathbf{z}) = \frac{1}{2}\sum_{i\in\mathcal{A}} z_i^2 \ . \tag{7}$$

The time derivative of V along the trajectories of (2) is

$$\begin{aligned}\dot{V} &= \sum_{i\in\mathcal{A}} z_i \dot{z}_i = \sum_{i\in\mathcal{A}} z_i\left[-\frac{w_i r_i}{B}[z_i - p_i]\right] \\ &- \sum_{i\in\mathcal{A}} c_i z_i \left[\sum_{k\in\mathcal{A}} \frac{r_k}{B}[z_i - z_k]\right] - \sum_{i\in\mathcal{A}} \frac{z_i^4}{r_i} \\ &\leq \sum_{i\in\mathcal{A}} \left[\alpha_i |z_i| + \beta_i |z_i|^2 - \gamma_i |z_i|^4\right]\end{aligned}$$

for some positive constants α_i, β_i and γ_i, for $i \in \mathcal{A}$. As the fourth-degree terms dominate the linear and quadratic terms for large enough $\mathbf{z}$, we see that $\dot{V}(\mathbf{z}) < 0$ for all $\mathbf{z}$ such that $\max_{i\in\mathcal{A}}\{|z_i|\} \geq \delta$ for some $\delta > 0$. This proves the claim of ultimate boundedness. ■

Note that one can explicitly find an ultimate bound η by computing the constants in the bound on $\dot{V}$ in the proof of Theorem 3.1. We skip it for brevity. Also note that Theorem 3.1 holds for all values of the parameters w_i, c_i, p_i and $r_i > 0$, $\forall i \in \mathcal{A}$. In the next subsection, we impose more conditions on the parameters and provide better results regarding the convergence of opinions. We also provide a more intuitive ultimate bound.

B. Stability and Convergence Analysis when all Agents are Non-Contrarian

This subsection focuses on the case where all agents are non-contrarian, *i.e.* $c_i \geq 0$, $\forall i \in \mathcal{A}$. We first analyze the stability of the equilibrium points of the dynamics (2) by linearizing the dynamics around them. This is summarized in the following result.

*Theorem 3.2: (**Stability of Equilibrium Points with All Non-Contrarian Agents**).* Suppose that $c_i \geq 0$, $\forall i \in \mathcal{A}$. Then, every equilibrium point, $\hat{\mathbf{z}} \in \mathcal{E}$, of the dynamics (2) is a locally asymptotically stable equilibrium point.

Proof: We prove the claim by linearizing the model around an arbitrary equilibrium point $\hat{\mathbf{z}} \in \mathcal{E}$. The ij^{th} element of $\mathbf{J}$, the Jacobian matrix evaluated at $\hat{\mathbf{z}}$, is

$$[\mathbf{J}]_{ij} = \begin{cases} \dfrac{-w_i r_i}{B} - c_i\left[1 - \dfrac{r_i}{B}\right] - \dfrac{3\hat{z}_i^2}{r_i}, & \text{if } i = j; \\ \dfrac{c_i r_j}{B}, & \text{if } i \neq j. \end{cases}$$

Notice that the i^{th} Gershgorin disc of the Jacobian matrix $\mathbf{J}$ has its center at

$$\frac{-w_i r_i}{B} - c_i\left[1 - \frac{r_i}{B}\right] - \frac{3\hat{z}_i^2}{r_i},$$

and has the radius as

$$\sum_{k\in\mathcal{A}\setminus\{i\}} \left|\frac{c_i r_k}{B}\right| = c_i\left[1 - \frac{r_i}{B}\right].$$

This is because $c_i \geq 0$, $\forall i \in \mathcal{A}$. Since $w_i, r_i > 0$, $\forall i \in \mathcal{A}$, all the Gershgorin discs are strictly contained in the negative half of the complex plane. Hence, $\mathbf{J}$ is Hurwitz and as a result $\hat{\mathbf{z}}$ is locally asymptotically stable. ■

Next, we show that if all the agents are non-contrarian, the solution of (2) starting from any initial condition converges to an equilibrium state.

*Theorem 3.3: (**Convergence of Opinions with All Non-Contrarian Agents**).* If $c_i \geq 0$, $\forall i \in \mathcal{A}$ then for any initial condition $\mathbf{z}(0) \in \mathbb{R}^n$ the solution $\mathbf{z}(t)$ of the opinion dynamics (2) or equivalently (3) converges asymptotically to an equilibrium point.

Proof: For this proof we use the form (3) of the opinion dynamics. The central idea of the proof is that the minimum and maximum of y_k's, denoted as

$$y_m := \min_{k\in\mathcal{A}}\{y_k\} \text{ and } y_M := \max_{k\in\mathcal{A}}\{y_k\},$$

converge to zero asymptotically. To show this, we first inspect the time derivative of y_i's, which we can derive from (3) as

$$\begin{aligned}\dot{y}_i &= -\frac{w_i r_i}{B} y_i - \frac{3z_i^2}{r_i} y_i + c_i\left(\sum_{k\in\mathcal{A}} \frac{r_k}{B} y_k - y_i\right) \\ &\leq -\left(\frac{w_i r_i}{B} + \frac{3z_i^2}{r_i}\right) y_i, \quad \forall i \text{ s.t. } y_i = y_M,\end{aligned}$$

since r_k's over all $k \in \mathcal{A}$ sum to B, $c_i \geq 0$ and $y_M \geq y_i$ for all $i \in \mathcal{A}$. Now, notice that the upper right hand derivative of y_M, $D^+ y_M$, satisfies

$$\begin{aligned}D^+ y_M &= \max\{\dot{y}_i \mid y_i = y_M,\ i \in \mathcal{A}\} \\ &\leq -\min_{i\in\mathcal{A}}\left\{\left(\frac{w_i r_i}{B} + \frac{3z_i^2}{r_i}\right)\right\} y_M,\end{aligned}$$

where in the equation, the maximization is over a variable set of agents at different times whereas in the inequality we fix the set of agents over which we minimize. From Theorem 3.1 and its proof, we know that given $\mathbf{z}(0)$, $\mathbf{z}(t)$ is lower and upper bounded for all time $t \geq 0$. As $w_i, r_i, B \in \mathbb{R}_{>0}$, we can use the comparison principle and say that there is an exponentially decaying bound on y_M. We can carry out parallel analysis on y_m also. In other words, $\exists\, a_m, a_M < 0$ and $\exists h_m, h_M > 0$ such that

$$-h_m \exp(a_m t) \leq y_m \leq y_k \leq y_M \leq h_M \exp(a_M t),\ \forall k \in \mathcal{A}.$$

Hence $|y_k|$ for each $k \in \mathcal{A}$ converges exponentially fast to zero. Thus, $\lim_{t\to\infty} z_k(t)$ also exists for each $k \in \mathcal{A}$. ■

To conclude this section, we give a better and more intuitive ultimate bound using the m_i's introduced earlier. Recall that for each $i \in \mathcal{A}$, $S_i(\cdot)$ has a unique root at m_i, *i.e.* $S_i(m_i) = 0$. First, we define

$$m_{\min} := \min\{m_i\}_{i\in\mathcal{A}}, \quad m_{\max} := \max\{m_i\}_{i\in\mathcal{A}} \tag{8}$$

and the corresponding interval

$$\mathcal{M} := [m_{\min}, m_{\max}]\,. \tag{9}$$

We are now ready to show that the solutions converge to $\mathcal{M}^n$. The proof of this result follows from Theorem 3.3.

Corollary 3.4: *(**Convergence to the set $\mathcal{M}^n$ when all Agents are Non-Contrarian**).* Suppose that $c_i \geq 0$, $\forall i \in \mathcal{A}$. Let $m_{\min}, m_{\max}$ and $\mathcal{M}$ be as defined in (8) and (9). Suppose $\mathbf{z}(t)$ is the solution to (2) from an initial condition $\mathbf{z}(0) \in \mathbb{R}^n$. Then $\mathbf{z}(t)$ converges to the set $\mathcal{M}^n$.

Further, suppose $m_{\min} < m_{\max}$. Then every equilibrium point $\hat{\mathbf{z}}$ of (2) lies in the interior of $\mathcal{M}^n$. Moreover $\exists\, T(\mathbf{z}(0)) \geq 0$ such that $\mathbf{z}(t) \in \mathcal{M}^n$, $\forall t \geq T(\mathbf{z}(0))$.

Proof: First note that $\forall\, i \in \mathcal{A}$,

$$S_i(z_i) \begin{cases} > 0, & z_i < m_i \\ = 0, & z_i = m_i \\ < 0, & z_i > m_i \end{cases}, \quad C_i(\mathbf{z}) \begin{cases} > 0, & z_i < \bar{z} \\ = 0, & z_i = \bar{z} \\ < 0, & z_i > \bar{z}. \end{cases} \tag{10}$$

Next, $\forall \mathbf{z} \notin \mathcal{M}^n$, it is easy to see from (10) that $\exists i \in \mathcal{A}$ such that $\dot{z}_i \neq 0$. Thus, if $\hat{\mathbf{z}} \in \mathcal{E}$ then $\hat{\mathbf{z}} \in \mathcal{M}^n$. Then by Theorem 3.3, the first claim is true.

Now, let $m_{\min} < m_{\max}$. From (10) notice that for any $\mathbf{z}^* \in \mathcal{E}$, for all $i \in \mathcal{A}$, $z_i^* \in [m_i, \bar{z}^*]$ or $z_i^* \in [\bar{z}^*, m_i]$ depending on the sign of $\bar{z}^* - m_i$. Further, $\bar{z}^*$ cannot be equal to $m_{\min}$ or $m_{\max}$ because that would mean $z_i^* = \bar{z}^*$ for all $i \in \mathcal{A}$ and (10) then means that there is at least one agent i for which $\dot{z}_i^* \neq 0$. Finally, $S_i(m_i) = 0$ and hence if $\bar{z}^*$ is in the interior of $\mathcal{M}$ then no agent i can have $z_i^* = m_{\min}$ or $z_i^* = m_{\max}$. Finite time convergence to $\mathcal{M}^n$ then follows from Theorem 3.3. This completes the proof. ∎

The bound proposed using $\mathcal{M}$ has the advantage that it depends only on the m_i's and hence can be computed easily using the parameters w_i, p_i and r_i. Moreover, these m_i's can be used to give a necessary and sufficient condition for existence of consensus equilibria, which we discuss next.

IV. Equilibrium Points of the Dynamics

In this section, we analyze the equilibrium points of the opinion dynamics model. Specifically, we provide conditions under which consensus and Nash equilibria can be equilibrium points of (2).

A. Consensus Equilibria

Here, we deal with the consensus equilibria of the model, i.e., equilibria of the form $\xi\mathbf{1}$, with $\xi \in \mathbb{R}$. We refer to the case of $\xi = 0$ as a neutral consensus since all the agents have neutral opinions in this case. On the other hand, we refer to the case of $\xi \neq 0$ as a non-neutral consensus. In the following lemma, we present conditions for (2) to have consensus equilibria. We use the form of the dynamics in (3) and the functions in (4) to justify our claims.

Lemma 4.1: *(**Necessary and Sufficient Conditions for Existence of Consensus Equilibrium**).* Consider the dynamics (2) and its equivalent representation (3). For each $i \in \mathcal{A}$, let $m_i \in \mathbb{R}$ be the unique point such that $S_i(m_i) = 0$. Then, $\xi\mathbf{1} \in \mathcal{E}$ if and only if $m_i = \xi$, $\forall i \in \mathcal{A}$. Further if $c_i \geq 0$, $\forall\, i \in \mathcal{A}$ and if $\exists\, M \in \mathbb{R}$ such that $m_i = M$, $\forall i \in \mathcal{A}$ then $\mathcal{E} = \{M\mathbf{1}\}$, i.e., $M\mathbf{1}$ is the unique equilibrium point of (2).

Proof: First, note that if $z_i = \xi$, $\forall i \in \mathcal{A}$, for some $\xi \in \mathbb{R}$ then $\bar{z} = \xi$, where $\bar{z}$ is defined in (5). Thus $C_i(\xi\mathbf{1}) = 0$, $\forall i \in \mathcal{A}$. Hence, from (3), $\xi\mathbf{1}$ is an equilibrium point iff $S_i(\xi) = 0, \forall i \in \mathcal{A}$. Since m_i is the unique root of $S_i(.)$ the first claim is true.

Next, consider the special case of $c_i \geq 0$, and $m_i = M$, $\forall\, i \in \mathcal{A}$ for some $M \in \mathbb{R}$. We show the claim about the unique equilibrium by contradiction. Suppose there exists an equilibrium point $\mathbf{z}^*$ such that $\mathbf{z}^* \neq M\mathbf{1}$. As $\bar{z}^*$ is a convex combination of z_i^*'s for $i \in \mathcal{A}$, there always exists an agent $i \in \mathcal{A}$ such that $z_i^* \leq \bar{z}^*$. Consider any $i \in \mathcal{A}$ such that $z_i^* \leq \bar{z}^*$ and $z_i^* < M$. Then, from (10), it is clear that $\dot{z}_i^* > 0$. Similar arguments can be given for an $i \in \mathcal{A}$ such that $z_i^* \geq \bar{z}^*$ and $z_i^* > M$ to show that $\dot{z}_i^* < 0$. This contradicts the assumption that $\mathbf{z}^*$ is an equilibrium point and hence completes the proof. ∎

Remark 4.2: *(**Consensus Formation among Agents**).* Lemma 4.1 states that it is both necessary and sufficient for all the m_i's to be the same for the opinion dynamics model to have a consensus equilibrium. It is evident that if the agents are to arrive at a consensus equilibrium, then all their preferences p_i's must be of the same sign. Further, if all the agents are non-contrarians and they have a consensus equilibrium, then it is the only equilibrium of the dynamics. In this case Theorems 3.2 and 3.3 imply that the agents always achieve consensus starting from any initial opinion vector. When $p_i = 0$, $\forall i \in \mathcal{A}$, then the only possible consensus equilibrium is the neutral consensus, i.e., every agent reaches a neutral opinion on the topic. If the preferences of the agents have different signs, then the opinions of agents can never reach an exact consensus in equilibrium. However, other equilibria that are arbitrarily close to consensus may still exist. •

When the agents attain a consensus equilibrium, we can measure how much influence an agent has on the whole group by measuring the deviation of the consensus value from its preference (*i.e.* if $\xi \in \mathbb{R}$ is the consensus value, we compute $|p_i - \xi|$ as a measure of the influence of the agent $i \in \mathcal{A}$). Note that if $p_i = 0$, for some $i \in \mathcal{A}$, then $m_i = 0$ and hence the only consensus equilibrium possible is neutral. So we consider $p_i \neq 0$, $\forall i \in \mathcal{A}$ to give the next result on dominance and discuss it in the remark following it.

Lemma 4.3: *(**Consensus Deviation from Preference**).* Consider the dynamics (2) or equivalently (3) and suppose $p_i \neq 0$, $\forall i \in \mathcal{A}$. For each agent $i \in \mathcal{A}$, define $\sigma_i := w_i r_i^2$ and $\Delta_i(\xi) := |p_i - \xi|$. Suppose $\xi\mathbf{1} \in \mathcal{E}$, with $\xi \in \mathbb{R}$. Then $\sigma_i \Delta_i(\xi) = \sigma_j \Delta_j(\xi)$, $\forall i, j \in \mathcal{A}$, and in particular $\Delta_i(\xi) < \Delta_j(\xi)$ if and only if $\sigma_i > \sigma_j$.

Proof: Since $p_i \neq 0$, $\forall i \in \mathcal{A}$, we also have $0 < |m_i| < |p_i|$ $\forall i \in \mathcal{A}$. Then by Lemma 4.1, $\xi \neq 0$. Further, from Lemma 4.1, we know that $r_i S_i(\xi) = 0$, $\forall i \in \mathcal{A}$, which again implies that

$$\sigma_i(p_i - \xi) = \sigma_j(p_j - \xi),\ \forall i, j \in \mathcal{A}.$$

Since $\sigma_i > 0$, $\forall i \in \mathcal{A}$, the result now follows. ∎

Remark 4.4: ***(Dominance in Consensus).*** Let us call the scalar $\sigma_i := w_i r_i^2$, $\forall i \in \mathcal{A}$ as the *dominance* weight of the agent i. If all agents have a non-neutral preference and the agents reach consensus equilibrium of (2) with a consensus value of $\xi \in \mathbb{R}$, then Lemma 4.3 states that, if an agent i has higher dominance weight than the agent j then the consensus value ξ is closer to agent i's preference than that of agent j. Note that the dominance weight is directly proportional to the weight the agent assigns to its preference and the square of the resources available to it. This means that an agent with very high resources can exert more influence even if it gives less weight to its internal preference. On the other hand, if an agent has lower resources, then it has to have much higher internal weight to have more influence in the group. •

Next, we explore opinion dynamics from a game theoretical point of view. In particular, we analyze the relationship between the equilibria of the dynamics and the Nash equilibria of the underlying game.

B. Nash Equilibria

Here we carry out a Nash Equilibrium analysis of the opinion formation game. Recall that every agent is interested in maximizing its utility U_i given in (1) by suitably choosing its opinion z_i. Thus, this can be thought of as a *strategic form game* $\mathcal{G} = \langle \mathcal{A}, (\mathbb{R})_{i\in\mathcal{A}}, (U_i)_{i\in\mathcal{A}} \rangle$ among the set of agents $\mathcal{A}$, with agent i's *strategy* being its opinion $z_i \in \mathbb{R}$ and its utility function being $U_i(\cdot)$. For the sake of convenience, we let z_{-i} denote the opinions of all agents other than i. Then, the set of *Nash equilibria* of the game $\mathcal{G}$ is

$$\mathcal{NE} := \{\mathbf{z}^* \in \mathbb{R}^n \mid \forall i \in \mathcal{A}, \\ U_i(z_i^*, z_{-i}^*) \geq U_i(z_i, z_{-i}^*), \forall z_i \in \mathbb{R}\}. \quad (11)$$

Note that for a Nash equilibrium $\mathbf{z}^*$, z_i^* is agent i's best response over all opinions $z_i \in \mathbb{R}$ to z_{-i}^*, the opinion profile of all the other agents. However, in the dynamics (2), each agent updates its opinion according to the gradient ascent of its utility with respect to its opinion while assuming that the other agents do not change their opinions. Hence, the agents at each time instant revise their opinion to only "local" best response. This motivates the next definition of a *local Nash equilibrium*.

Definition 4.5: ***(Local Nash Equilibrium).*** A strategy profile $\mathbf{z}^* \in \mathbb{R}^n$ is said to be a local Nash equilibrium if and only if $\forall i \in \mathcal{A}$, $\exists\, \rho_i \in \mathbb{R}_{>0}$ such that

$$U_i(z_i^*, z_{-i}^*) \geq U_i(z_i, z_{-i}^*),\ \forall\, z_i \text{ s.t. } |z_i^* - z_i| \leq \rho_i\,. \quad (12)$$

Let the set of local Nash equilibria of $\mathcal{G}$ be denoted by $\mathcal{NE}_l$. •

It is easy to see that a Nash equilibrium is also a local Nash equilibrium and hence $\mathcal{NE} \subseteq \mathcal{NE}_l$. In the next result, we show that every local Nash equilibrium is an equilibrium of the opinion dynamics.

Lemma 4.6: ***(Local Nash Equilibrium is an Equilibrium of the Dynamics in (2)).*** If an opinion profile $\mathbf{z}^*$ is such that $\mathbf{z}^* \in \mathcal{NE}_l$ then $\mathbf{z}^* \in \mathcal{E}$, with $\mathcal{E}$ defined in (6).

Proof: Since $\mathbf{z}^* \in \mathcal{NE}_l$, it implies that $\forall i \in \mathcal{A}$, z_i^* locally maximizes $U_i(\cdot, z_{-i}^*)$. Thus, the partial derivative of $U_i(\cdot)$ with respect to z_i evaluated at z_i^* is *zero*. The claim then follows immediately from (2) and (6). ■

Lemma 4.6 states that every local Nash equilibrium of $\mathcal{G}$ is also an equilibrium point of dynamics (2). But the converse need not be true. In the following result, we give conditions for an opinion profile $\mathbf{z}^* \in \mathbb{R}^n$ that is an equilibrium point of the opinion dynamics model to be a local Nash equilibrium of the opinion formation game.

Theorem 4.7: ***(Conditions for an Equilibrium Point of (2) to be a Local Nash Equilibrium).*** Consider the dynamics (2) and the set of equilibrium points $\mathcal{E}$ in (6). Let $\mathcal{G} = \langle \mathcal{A}, (\mathbb{R})_{i\in\mathcal{A}}, (U_i)_{i\in\mathcal{A}} \rangle$ represent the corresponding strategic form game of opinion formation. Suppose $\mathbf{z}^* = (z_i^*, z_{-i}^*) \in \mathcal{E}$. For each agent $i \in \mathcal{A}$, define

$$\tau_i := \frac{1}{3}\left[-c_i r_i\left(1 - \frac{r_i}{B}\right) - \frac{w_i r_i^2}{B}\right]. \quad (13)$$

Then, $\mathbf{z}^* \in \mathcal{NE}_l$ only if

$$(z_i^*)^2 \geq \tau_i,\ \forall i \in \mathcal{A}\,. \quad (14)$$

Moreover, if the inequality in (14) is strict, then $\mathbf{z}^* \in \mathcal{NE}_l$.

Proof: Let the hypothesis be true. From Definition 4.5 we know that $\mathbf{z}^* = (z_i^*, z_{-i}^*)$ is a local Nash equilibrium if and only if $\forall i \in \mathcal{A}$, z_i^* locally maximizes $U_i(\cdot, z_{-i}^*)$. Now since $\mathbf{z}^* \in \mathcal{E}$, by the definitions in (2) and (6), it is clear that for each $i \in \mathcal{A}$, z_i^* satisfies the first-order necessary conditions for a local maximizer.

Now suppose that $\mathbf{z}^* \in \mathcal{NE}_l$. Then we have that

$$\left.\frac{\partial^2}{\partial z_i^2} U_i(z_i, z_{-i}^*)\right|_{z_i^*} = \frac{-w_i r_i}{B} - c_i\left(1 - \frac{r_i}{B}\right) - 3\frac{(z_i^*)^2}{r_i} \\ \leq 0, \quad \forall i \in \mathcal{A}\,, \quad (15)$$

This proves the necessary condition in (14).

Finally, note that if the inequality in (15) is strict for a $\mathbf{z}^* \in \mathcal{E}$, then by the second order sufficiency condition of optimality, z_i^* is a local maximizer of $U_i(z_i, z_{-i}^*)$ for each $i \in \mathcal{A}$. This completes the proof. ■

The statement of the previous result can be combined with the result in Theorem 3.1 to provide a condition for which the opinion formation game does not have any local Nash equilibrium. We state this in the next result, proof of which is intuitive since no equilibrium point of (2) can exist beyond any ultimate bound (which always exists).

Corollary 4.8: ***(Non-Existence of Local Nash Equilibria).*** Suppose η is an ultimate bound for the dynamics in (2). If there exists an agent $i \in \mathcal{A}$ such that $\tau_i > \eta$, with τ_i defined in (13), then $\mathcal{NE}_l = \varnothing$. ■

Finally, to end this section, we revisit the case of non-contrarian agents and specialize the above results for the case when $c_i \geq 0$, $\forall i \in \mathcal{A}$. We state this in the following result.

Theorem 4.9: ***(Nash Equilibria when all Agents are Non-Contrarian).*** Consider the dynamics in (2), the set of equilibrium points $\mathcal{E}$ in (6) and the set of local Nash equilibria defined in Definition 4.5. Suppose $c_i \geq 0, \forall i \in \mathcal{A}$. Then the following statements are equivalent.

(i) $\mathbf{z}^* \in \mathcal{E}$, (ii) $\mathbf{z}^* \in \mathcal{NE}_l$, (iii) $\mathbf{z}^* \in \mathcal{NE}$.

Proof: First note that (iii) $\implies$ (ii) is trivially true and (ii) $\implies$ (i) follows from Lemma 4.6. Next, we show that (i) $\implies$ (ii). Notice that $\forall i \in \mathcal{A}$, $c_i \geq 0$ (by hypothesis), $w_i, r_i > 0$ and $B = \sum_{j\in\mathcal{A}} r_j$. Then it is easy to see from (13) that $\tau_i < 0$, $\forall i \in \mathcal{A}$. Then Theorem 4.7 proves the implication.

Finally, we show that (ii) $\implies$ (iii). Notice that if $c_i \geq 0$, $\forall i \in \mathcal{A}$ then (15) holds with strict inequality. This means that $\forall i \in \mathcal{A}$, $U_i(\cdot, z^*_{-i})$ is a strictly concave function for each z^*_{-i}. Then from the definition of $\mathcal{NE}$ and $\mathcal{NE}_l$, the implication is true. This completes the proof. ■

V. Simulations

In this section, we demonstrate our results using simulations. We have used MATLAB and the ODE45 solver for simulations. We consider a group of 6 non-contrarian agents forming their opinions according to (2). In Figure 1, we study a case where the opinions of all agents reach a non-neutral consensus equilibrium with a consensus value equal to 35. The model parameters [1] used to simulate this case are as follows, the vector containing the initial opinions of all six agents is $\mathbf{z}_0 = \begin{bmatrix}40.54 & -69.28 & 90.69 & 8.18 & 35.95 & -92.69\end{bmatrix}$, the vector containing the conformity weights of all six agents is $\mathbf{c} = \begin{bmatrix}14.87 & 2.12 & 13.63 & 9.27 & 4.24 & 1.97\end{bmatrix}$, the vector containing the weights w_i of all six agents is $\mathbf{w} = \begin{bmatrix}18.71 & 9.16 & 4.81 & 15.28 & 15.19 & 14.81\end{bmatrix}$, the vector containing the resources of all six agents is $\mathbf{r} = \begin{bmatrix}8.2357 & 1.7501 & 1.6357 & 6.6599 & 8.9439 & 5.1656\end{bmatrix} \times 10^3$ and $m_i = 35$, $\forall i \in \mathcal{A}$ which is equal to the consensus value. Thus, the necessary and sufficient condition of Lemma 4.1 is satisfied. The preference vector containing preferences (p_i) of all six agents is, $\mathbf{p} = \begin{bmatrix}36.09 & 84.51 & 142.92 & 37.05 & 36.14 & 38.51\end{bmatrix}$. For every agent $i \in \mathcal{A}$, let z_i^∞ denote its asymptotic opinion value. The vector whose each element is the absolute difference between an agent's final consensus opinion and its preference opinion $(|z_i^\infty - p_i|)$ is $\mathbf{d} = \begin{bmatrix}1.09 & 49.51 & 107.93 & 2.05 & 1.14 & 3.51\end{bmatrix}$. The vector containing the dominance weights of all six agents is $\boldsymbol{\sigma} = \begin{bmatrix}1.269 & 0.028 & 0.0128 & 0.677 & 1.214 & 0.395\end{bmatrix} \times 10^9$. From this data, we can verify that the dominance claim in Lemma 4.3 is satisfied in this case.

In Figure 2, we study a case where all agents are non-contrarians and do not achieve consensus. The model parameters for all six agents in this case are, $\mathbf{p} = \begin{bmatrix}45.04 & -54.02 & 15.21 & 62.13 & -19.23 & 97.69\end{bmatrix}$, $\mathbf{z}_0 = \begin{bmatrix}-82.0 & -35.81 & 2.28 & -87.88 & 45.14 & 11.31\end{bmatrix}$ $\mathbf{w} = \begin{bmatrix}12.85 & 4.43 & 16.74 & 19.42 & 16.93 & 10.12\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}5.58 & 14.93 & 4.74 & 19.15 & 12.41 & 12.01\end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix}1.726 & 0.9035 & 2.5526 & 8.5857 & 9.1107 & 6.9963\end{bmatrix} \times 10^3$.

[1]For the entirety of Section V, the i^{th} element of any parameter data vector corresponds to agent i. Further, all numbers have been rounded off to two decimal places.

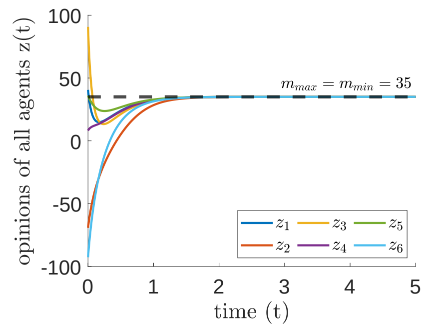

Fig. 1: A group of 6 non-contrarian agents attaining a non-neutral consensus equilibrium with consensus value= 35.

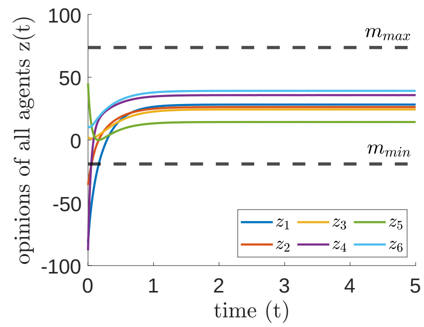

Fig. 2: A group of 6 non-contrarian agents not reaching consensus but converging to an equilibrium within the ultimate bound $\mathcal{M}^n$.

The agents do not achieve consensus because $\mathbf{m} = \begin{bmatrix}27.97 & -16.55 & 14.39 & 58.05 & -19.08 & 73.62\end{bmatrix}$ which violates the necessary condition for consensus given in Lemma 4.1. From Figure 2, it can also be seen that the opinion profile converges to an equilibrium. Moreover, the opinions of all agents enter the set $\mathcal{M} = [-19.08, 73.62]$ in finite time, which is consistent with the result of Corollary 3.4.

When the conformity weights are allowed to be negative, we found using simulations that the opinions evolving according to our model can exhibit a periodic or oscillatory behavior. Figure 3 depicts a scenario where the opinions of two agents, having conformity weights equal to -10 and +10, evolve according to (2) and exhibit a periodic behavior. The model parameters for this case are $\mathbf{z_0} = \begin{bmatrix}25.05 & 8.61\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}-10 & 10\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}3.78 & 0.02\end{bmatrix}$, $\mathbf{p} = \begin{bmatrix}10.27 & 18.66\end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix}3164.2 & 6996.17\end{bmatrix}$. We do not have any analytical result which explains such behavior. The analysis and study of limit cycles for the model are left for future works.

VI. Conclusions

In this paper, we have introduced a model which captures the effect of resources on the opinion formation process. Extreme opinions are penalized by the "limited" resources of the agents. We have proved boundedness of opinions for our model. For the case when all agents are non-contrarians, we have proved the stability of equilibria and convergence of opinions to some equilibrium. For a consensus equilibrium, we justified that the greater the dominance weight of an

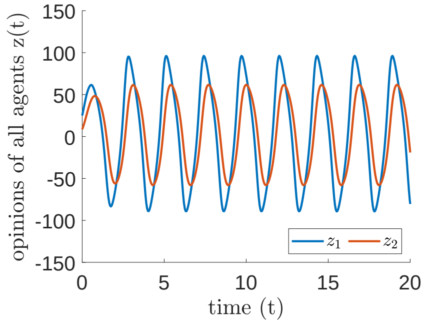

Fig. 3: Periodic behavior exhibited by opinions of two agents having the conformity weights $c_1 = -10$ and $c_2 = 10$.

agent, the smaller the deviation of the consensus value from its internal preference. We analyzed our model from a game theoretic perspective and provided results for an equilibrium to be a local Nash equilibrium of the opinion formation game. When all agents are non-contrarian, we also showed that the set of equilibria of the opinion dynamics is the same as the set of local Nash equilibria and Nash equilibria of the underlying game. Future work directions include extensions of the above model to a multi-topic scenario, including a social network, analyzing periodic behavior of opinions in the presence of contrarians, quantized consensus, the relationship between resources and social power, and exploration of more general resource penalty functions.

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