On Co-design of Event Trigger and Quantizer for Emulation Based Control

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Abstract— **In this paper the inter-dependence of quantization and event-triggered control is investigated. We motivate the idea of co-designing the event-trigger and the quantizer for emulation based discrete-event control, and then propose a methodology for designing emulation based discrete-event controllers for asymptotic stabilization of general nonlinear systems. The proposed algorithm results in an easily implementable finite density logarithmic quantizer and a simple event-trigger. The resulting emulation based discrete-event controller semi-globally asymptotically stabilizes the origin of the system with a specified arbitrary compact region of attraction. The quantizer is designed to be endowed with hysteresis so as to avoid chattering of the controller. The proposed design is illustrated with a two-dimensional nonlinear system.**

I. Introduction

A common challenge in Cyber Physical Systems (CPS) is control under data rate constraints and limited computational capabilities, a problem that has been actively researched in the last decade. Many papers have looked at issues such as fundamental limits on communication rate for stabilization, while others have focused on asymptotic stabilization with dynamic quantization. A good survey paper on these and other topics is [1]. In this paper, we focus on the problem of control under sampling as well as quantization.

The newly emerging field of event-triggered control (example: [2], [3], [4], [5]), seeks to systematically design controllers that update or sample the control action at low average rates. These controllers are based on the principle of updating the control only when necessary (control by exception). In other words, event-triggered control seeks to minimize the average rate of communication instances, although the amount of information that can be conveyed at each communication instance is not limited. However, in practical situations quantization is inevitable, and hence it is necessary to consider event based control along with quantization. The survey paper [6] makes a related remark that the connection between quantized and even-triggered feedback must be studied. Keeping in mind the limited computational resources, in this paper we only consider static quantizers.

The above discussion motivates the need for design of coarsest static quantizers. Elia and his co-workers first studied this problem in the context of quadratically stabilizable linear time invariant systems [7], [8] (single input), [9] (two input), and demonstrated that the coarsest quantizer is the logarithmic quantizer. Fu and Xie [10] extended the results of [8] to linear multiple input systems by quantizing each dimension separately, and their design resulted in an infinite-density logarithmic quantizer. Finite density logarithmic quantizers for the multiple input case were designed in [11], [12] (and references therein). All the above references focussed on Linear Time Invariant (LTI) systems and, except for [7], [8], the results were only developed for discrete time systems. While [7] designed an implicitly verified discrete-event controller, [8] studied the optimal periodic sampling time. The references [13], [14] utilized a Robust Control Lyapunov Function (RCLF) approach to characterize the coarsest quantizers for single input control affine nonlinear systems.

Motivation: Systems with quantization can be viewed as switched systems [15], the switching surfaces being the boundaries of the quantization cells. In other words, a quantizer is a discrete-event encoder, whose output is the quantization state. The quantization state evolves in a discrete set and the boundaries of the quantization cells determines the event-trigger. The complexity of the event-triggering condition is determined by the complexity of the shape of the quantization cells. An RCLF approach to quantization in nonlinear systems may lead to very complicated geometries (for example, see Equation (10) in [13]) and the event-triggering condition may be as computationally intensive, if not more, as the original control law.

Thus, we see that on the one hand, event-triggered control [2], [3], [4], [5] assumes the availability of an infinite precision quantizer and on the other an RCLF quantizer assumes that the induced event-trigger is computationally inexpensive. Therefore, in the context of Cyber Physical Systems, there is a need to co-design the quantizer and the event-trigger for emulation based control.

Contributions: In this paper we exploit the common principle behind event based control (as in [2], [5]) and coarsest quantization, which is robustness to measurement errors, to design a discrete-event controller for semi-global asymptotic stabilization of general nonlinear systems. Specifically, we propose a methodology for co-designing the event-trigger and the quantizer in an emulation based controller. Although the resultant quantizer is not necessarily coarsest, it is however a finite density logarithmic quantizer and is easy to implement. The proposed algorithm produces an implicitly verified emulation based discrete-event controller

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that asymptotically stabilizes the origin with a specified arbitrary compact region of attraction. In the special case that a certain Lipschitz constant holds globally, the origin of the closed loop system is globally asymptotically stabilizable. In comparison to the coarsest quantization literature, our quantizer design holds for general multi-input nonlinear continuous time systems. Another important aspect of the proposed quantizer is the presence of hysteresis, which is utilized for guaranteeing a dwell time for the updates of the discrete event controller.

The rest of the paper is organized as follows. Section II introduces the basic notation and states precisely the problem under study. The design of the event-trigger is discussed in Section III and the quantizer design is described in Section IV. An example of a two dimensional nonlinear system is provided in Section V and finally some concluding remarks are made in Section VI.

II. Problem statement

Note: The results in Sections II and III do not depend on a specific choice of a norm. However, the proposed quantizer design utilizes the max or the infinity norm. Therefore, we adopt this norm through out the paper, and use the notation $|y|$ to denote the max norm, $\|y\|_\infty$, of a vector y.

Consider a nonlinear system of the form

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{1}$$

with feedback control $u = \kappa(x)$ that renders the origin of the closed loop system

$$\dot{x} = f(x,\kappa(x)) \tag{2}$$

globally asymptotically stable. Now, consider the problem of controlling the system with quantized state feedback, where the quantizer is static. A static quantizer can be modeled as a nonlinear function of the state. However, in this paper we consider quantizers with hysteresis (hence memory). Thus, we define the quantizer function in a more general sense as follows.

Definition 1: A *quantizer* is a function $q : \mathbb{R}^n \times \Omega \longrightarrow \Omega$, where $\Omega = \{\omega^0, \omega^1, \omega^2, \dots\}$ is a countable set, with $\omega^k \in \mathbb{R}^n$ for each k and $\bigcup_{\omega^k \in \Omega} \{x \in \mathbb{R}^n : q(x,\omega^k) = \omega^k\} = \mathbb{R}^n$.

In this paper ω^k are called the *generating points* and Ω is called the *generating set* (or the *set of generating points*) of the quantizer. The quantization density is defined as

Definition 2: Quantization density: For $0 < \epsilon \le 1$, let $N(\epsilon)$ be the number of elements $\omega \in \Omega$ such that $\epsilon \le |\omega| \le 1/\epsilon$. The quantization density of the quantizer q is defined as

$$\eta_q = \limsup_{\epsilon \to 0} \frac{N(\epsilon)}{-2\ln(\epsilon)}. \tag{3}$$

This definition is similar to the one in [8].

The presence of hysteresis in the quantization state x_q, and the interpretation of a quantizer as a discrete-event encoder necessitates the treatment of x_q as a state variable and the resultant closed loop system as a hybrid system. In this paper, we adopt the notation and theory developed by Teel and his co-workers (see [16] and the references therein) to study this hybrid system. Let $\xi = [x; x_q] \in \mathbb{R}^n \times \Omega$ denote the state of the hybrid system (the notation $[x; x_q]$ denotes the concatenation of the vectors x and x_q). Then, the closed loop hybrid system may be expressed as

$$\dot{\xi} = F(\xi) := \begin{cases} \dot{x} = f(x,\kappa(x_q)) \\ \dot{x}_q = 0 \end{cases}, \quad \xi \in C \tag{4}$$

$$\xi^+ = G(\xi) := \begin{cases} x^+ = x \\ x_q^+ = q(x,x_q) \end{cases}, \quad \xi \in D \tag{5}$$

$$\mathcal{H} = (C,F,D,G) \tag{6}$$

where $C \subset \mathbb{R}^n \times \Omega$ and $D \subset \mathbb{R}^n \times \Omega$ are appropriately defined sets. The hybrid system $\mathcal{H}$ is the collection of the flow set, C, the flow map, F, the jump set, D, and the jump map, G. The quantizer is specified by the set Ω and the function $q(x,x_q)$. As is clear from our formulation, in this paper the updates of the quantized state information, x_q, are not periodic, unlike in [8]. Rather, the quantized state is updated whenever a state-dependent triggering condition is satisfied, that is when $\xi \in D$.

The event-trigger determines **when** the feedback is communicated and the control updated. The quantizer determines **what** is communicated. As discussed earlier, an efficient discrete event controller necessitates the co-design of the event-trigger and the quantizer. Therefore, the problem under consideration in this paper is that of co-designing an event-trigger and a quantizer in emulation based controllers for semi-global asymptotic stability of general nonlinear systems. Specifically, the problem is to design the sets Ω, C and D; and the quantizer function q such that both the event-trigger and the quantizer are efficient. In the next section, the design of the event-trigger (design of the sets C and D) is detailed.

III. Design of the Flow and the Jump Sets

The following are the main assumptions in the paper.

(A1) The closed loop system (2) is input-to-state stable (ISS) with respect to measurement error, i.e., there exists a C^1 Lyapunov function, $V : \mathbb{R}^n \to \mathbb{R}$, that satisfies

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$$
$$\frac{\partial V}{\partial x} f(x,\kappa(x+e)) \le -\alpha(|x|), \quad \text{if } \gamma(|e|) \le |x|$$

where $\alpha_1(.)$, $\alpha_2(.)$, $\alpha(.)$ and $\gamma(.)$ are class $\mathcal{K}_\infty$[1] functions.

(A2) The function γ is Lipschitz on compact sets.

The relatively strong ISS assumption is not necessary for the proposed design. It is sufficient to assume that the origin of the system (2) is asymptotically stable. This weaker assumption is not pursued in this paper due to space constraints.

Define the measurement/quantization error as,

$$e \triangleq x_q - x. \tag{7}$$

[1] A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is said to belong to the class $\mathcal{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ [17].

and let us define the flow and the jump sets as

$$C = \{\xi \in \mathbb{R}^n \times \Omega : |e| \leq W|x|\} \tag{8}$$
$$D = \{\xi \in \mathbb{R}^n \times \Omega : |e| \geq W|x|\} \tag{9}$$

where W is a positive constant. The sets C and D capture a simple event-triggering condition.

A hybrid Lyapunov function candidate [16] is defined as follows.

Definition 3 (Lyapunov-function candidate): Given the hybrid system $\mathcal{H}$ with data (C, F, D, G) and the compact set $\mathcal{A} \subset \mathbb{R}^p$, the function $V_h : \text{dom}\, V_h \to \mathbb{R}$ is a *Lyapunov-function candidate* for $(\mathcal{H}, \mathcal{A})$ if (i) V_h is continous and nonnegative on $(C \cup D) \setminus \mathcal{A} \subset \text{dom}\, V_h$, (ii) V_h is continuously differentiable on an open set $\mathcal{O}$ satisfying $C \setminus \mathcal{A} \subset \mathcal{O} \subset \text{dom}\, V_h$, and (iii) $\lim_{\{\xi \to \mathcal{A}, \xi \in (\text{dom}\, V_h) \cap (C \cup D)\}} V_h(\xi) = 0$.

Let

$$\mathcal{A} \triangleq \{\xi \in \mathbb{R}^n \times \Omega : x = x_q = 0\} \tag{10}$$

and define the hybrid Lyapunov function candidate for the pair $(\mathcal{H}, \mathcal{A})$ as

$$V_h(\xi) = V(x) + \max\{0, |x_q - x| - 2W|x|\} \tag{11}$$

Notice that $V_h(\xi) = V(x)$ for all $\xi \in C$. The function $V_h(\xi)$ is positive definite and its sub-level sets are compact. Also note that $\langle \nabla V_h(\xi), F(\xi) \rangle = \frac{\partial V}{\partial x} f(x, \kappa(x_q))$ for all $\xi \in C \setminus \mathcal{A}$ and in an open neighborhood of $C \setminus \mathcal{A}$.

A. Selection of W

Let

$$B_r = \{x \in \mathbb{R}^n : |x| \leq r\}. \tag{12}$$
$$E_r = \{\xi \in \mathbb{R}^n \times \Omega : |x| \leq r,\ |x_q| \leq r\}. \tag{13}$$

Note that for each r finite, B_r and E_r are compact sets in $\mathbb{R}^n$ and $\mathbb{R}^n \times \Omega$, respectively. For each $\mu \geq 0$ define

$$\mathcal{R} \triangleq \{\xi : \mathbb{R}^n \times \Omega : V(x) \leq \mu,\ |x_q| \leq R \triangleq \alpha_1^{-1}(\mu)\} \tag{14}$$

where $\alpha_1(.)$ is the function from assumption (A1). Then, it is clear that $\mathcal{R} \subset E_R$. For each compact set $\mathcal{B}$ that contains the origin, there is a $\mu \geq 0$ such that $\mathcal{B} \subset \mathcal{R}$. Therefore, without loss of generality it is assumed that the prescribed region of attraction is of the form (14). If assumption (A2) holds, then there exists a constant $W_R > 0$ such that

$$W_R|x| \leq \gamma^{-1}(|x|), \quad \forall x \in B_R$$

The design of the flow and the jump sets is complete if we specify how the constant W is to be chosen. The following Lemma provides a methodology for accomplishing this goal.

Lemma 1: Consider the hybrid system (6) with C and D defined as in (8)-(9). Suppose assumptions (A1) and (A2) hold. Let the desired region of attraction be $\mathcal{R}$, (14), for some $\mu \geq 0$. If $W \leq W_R$ then

$$\langle \nabla V_h(\xi), F(\xi) \rangle < 0,\ \forall \xi \in C \cap \mathcal{R} \setminus \mathcal{A} \tag{15}$$

Proof: By the definition of W_R and the fact that $W \leq W_R$, it follows that

$$W|x| \leq W_R|x| \leq \gamma^{-1}(|x|), \quad \forall x \in B_R$$

Recall the definition of the flow set C, (8). Also, $\mathcal{R}$ is a subset of E_R, (13). Therefore, $(C \cap \mathcal{R}) \subset \{\xi \in \mathbb{R}^n \times \Omega : |e| \leq \gamma^{-1}(|x|)\}$ and assumption (A1) immediately implies that (15) is true. ■

Remark 1: If the function γ is globally Lipschitz, then (15) holds for all $\xi \in C \setminus \mathcal{A}$ and not just for $\xi \in (C \cap \mathcal{R}) \setminus \mathcal{A}$.

If $W_R \geq 1$ then quantization is not required and a constant control $u \equiv \kappa(0)$ asymptotically stabilizes the origin of the nonlinear system (1). This is made more precise in the following proposition and the subsequent discussion.

Proposition 1: Consider the hybrid system (6) with C and D defined as in (8)-(9). If $W_R \geq W > 1$ and $\Omega = \{0\}$, then the set $\mathcal{A}$, (10), is asymptotically stable with $\mathcal{R}$ included in the region of attraction.

Proof: The set $(D \cap \mathcal{R}) = \{x \in \mathbb{R}^n \times \{0\} : |x - 0| \geq W|x|\} \cap \mathcal{R} = \emptyset$, the empty set. On the other hand, $(C \cap \mathcal{R}) = \{x \in \mathbb{R}^n \times \{0\} : |x - 0| \leq W|x|\} \cap \mathcal{R} = \mathcal{R}$. Lemma 1 then implies that the set $\mathcal{A}$ is asymptotically stable with $\mathcal{R}$ included in the region of attraction. ■

If $W_R \geq W > 1$ and $\Omega = \{0\}$, then the set D is empty. Thus, the hybrid system (6) is really just the continuous time system (4) with $x_q \equiv 0$. If $W_R \geq W = 1$ and $\Omega = \{0\}$, then $(C \cap \mathcal{R}) = (D \cap \mathcal{R}) = \mathcal{R}$ and there can be jumps in the solutions of $\mathcal{H}$, (6). However, the jump map is the trivial map, $x^+ = x$ and $x_q^+ = x_q = 0$. Since the jump map is induced by the controller and is not inherent in the system, the trivial map can be ignored by the controller and we can focus only on purely flowing solutions that start in $\mathcal{R} \setminus \mathcal{A}$. All such solutions asymptotically converge to the set $\mathcal{A}$.

Therefore, in the sequel we assume that $W = W_R < 1$ unless specifically mentioned otherwise. In the next section the design process of the quantizer is detailed.

IV. Design of The Quantizer

Now, all that is left to be designed is the quantizer. Our goal here is the following. Given an event-trigger, (8)-(9), satisfying Lemma 1, design an efficient quantizer that semi-globally asymptotically stabilizes the origin of the system with a prescribed compact region of attraction.

In the coarsest quantizer literature, robustness to measurement errors is exploited to design finite density logarithmic quantizers and in single input LTI systems the coarsest quantizer. The quantizer in this paper also utilizes the same principle, although indirectly through the simplified event triggering condition designed in Section III. In our opinion, this approach is better suited for continuous time nonlinear systems for two reasons. Considering general nonlinear systems, the set $\{x \in \mathbb{R}^n : \frac{\partial V}{\partial x} f(x, \kappa(x_q)) < 0\}$ for an arbitrary x_q can have a complicated shape. This can potentially lead to a complex design process, that requires significant customization for individual systems. The second drawback is that of implementation - checking, in real time, whether the state belongs to a particular quantization cell can be as intensive, if not more, as computing the control itself. This defeats our motivation of designing controllers that require low rate of communication and low computation capabilities.

The proposed quantizer design is much simpler and applicable to a wide range of nonlinear systems. The chief features of the proposed quantizer design are as follows. The quantization cells are determined by the simplified triggering condition, (8)-(9). In the triggering condition, the *max or the infinity norm* is used, leading to a very easily implementable triggering condition and quantizer. The quantization cells are allowed to be overlapped, and the resulting hysteresis is utilized to avoid chattering of the controller.

Definition 4: For each $k \in \{0, 1, 2, \ldots\}$, the *quantization cell* generated by ω^k is the set $C^k = \{x \in \mathbb{R}^n : q(x, \omega^k) = \omega^k\}$.

In the hybrid system (6), x_q changes only during jumps. In order to minimize the number of control updates or jumps, it is necessary to ensure that at each jump the state is mapped outside the jump set D, or more precisely, it is required that

$$\xi^+ = G(\xi) \in (C \setminus D) \cap E_R, \quad \forall \xi \in (D \cap E_R) \setminus \mathcal{A} \tag{16}$$

However, x does not change during jumps, and $x_q^+ = q(x, x_q)$. Hence, the quantizer needs to be designed such that $x_q^+ \neq x_q$. Therefore, by the definition of a quantization cell it is necessary that $\left(C^k \times \{\omega^k\}\right) \cap (D \cap E_R) \setminus \mathcal{A} = \emptyset$ for each $k \in \{0, 1, 2, \ldots\}$. In other words, the quantizer must be defined such that for each $k \in \{0, 1, 2, \ldots\}$

$$\left(C^k \times \{\omega^k\}\right) \cap (C \cap E_R) = (C^k \times \{\omega^k\}) \cap E_R \tag{17}$$

Finally, $\mathcal{A} \subset C$, $\mathcal{A} \subset D$ and $\mathcal{A} \cap C^k = \emptyset$ if $\omega^k \neq 0$. Hence, it is necessary to choose $\omega^0 = 0$. Therefore, the quantizer has to satisfy the following constraints.

$$\omega^k \in \mathbb{R}^n \text{ and } |\omega^k| \leq R, \; k \in \{1, 2, \ldots\} \tag{18}$$

$$C^k = \{x \in \mathbb{R}^n : |\omega^k - x| < W_R|x|\}, \, k \in \{1, 2, \ldots\} \tag{19}$$

$$\omega^0 = 0 \tag{20}$$

$$C^0 = \{0\} \cup \{x \in \mathbb{R}^n : |x| > R\} \tag{21}$$

$$C^0 \cup \left(\bigcup_{k=1}^{k=\infty} \overline{C_\rho^k} \right) = \mathbb{R}^n, \quad 0 < \rho < 1 \tag{22}$$

where $C_\rho^k = \{x \in \mathbb{R}^n : |\omega^k - x| \leq \rho W_R|x|\}$ and $\overline{C_\rho^k}$ denotes the closure of the set C_ρ^k. Note that $C_\rho^k \subset C^k$ for each k. The constraint (22) has been introduced so that the resultant quantizer is over designed and the quantization cells overlap. In other words, the final constraint induces hysteresis in the quantizer, which is useful for avoiding chattering. Moreover, excluding C^0, each cell C^k is such that in the region where C^k overlaps with no other cell, $|\omega^k - x| \leq \rho W_R|x|$. Next, notice that the cell C^0 includes the region outside B_R. Any arbitrary nominal value could have been chosen as the quantization state for the region outside B_R, and we have selected it to be 0.

We define the quantizer function as follows.

$$q(x, \omega^k) = \begin{cases} \omega^k, & \text{if } x \in C^k \\ \underset{\omega^j}{\operatorname{argmin}} \dfrac{|\omega^j - x|}{W_R|x|}, & \text{if } x \notin C^k, \; x \neq 0 \\ \omega^0, & \text{if } x = 0 \end{cases} \tag{23}$$

In the second case there can be more than one solution. Note that the quantizer function satisfies (16).

The following theorem demonstrates that a quantizer that satisfies (18)-(23) asymptotically stabilizes the set $\mathcal{A}$ with $\mathcal{R}$ in the region of attraction.

Theorem 1: Consider the hybrid system (6) with C and D defined as in (8)-(9), and suppose assumptions (A1) and (A2) hold. Let the desired region of attraction be $\mathcal{R}$, (14), for some $\mu \geq 0$. Suppose that $W \leq W_R$ and that the quantizer is designed to satisfy (18)-(23). Then, the set $\mathcal{A}$ is asymptotically stable and the region of attraction includes $\mathcal{R}$.

Proof: The compact set $\mathcal{R} \subset E_R$, where $R = \alpha_1^{-1}(\mu)$. The function V_h in (11) is a hybrid Lyapunov candidate function for the pair $(\mathcal{H}, \mathcal{A})$. Consider the event-trigger (the sets C and D) designed in (8)-(9). Given a quantizer that satisfies (18)-(23), the following hold.

$$\langle \nabla V_h(\xi), F(\xi) \rangle < 0, \; \forall \, \xi \in C \cap \mathcal{R} \setminus \mathcal{A}$$
$$V_h(G(\xi)) - V_h(\xi) \leq 0, \; \forall \, \xi \in D \cap \mathcal{R} \setminus \mathcal{A}$$

where the first relation follows from Lemma 1, and the second from the fact that the quantizer function q ensures satisfiability of (16). Hence, for every $c > 0$ no complete solution remains in the compact set $\{\xi \in \mathbb{R}^n \times \Omega : V_h(\xi) = c\} \cap \mathcal{R}$. Recall the definition of $\mathcal{R}$, (14). The function $V(x)$ decreases monotonously during flows and does not change during jumps. The constraints (18) and (20) imply that $|x_q| \leq R$ at all times. Hence, $\mathcal{R}$ is forward-invariant[2] and every maximal solution that starts in $\mathcal{R}$ is a complete solution. Therefore, by Theorem 23 in [16] we conclude that the set $\mathcal{A}$ is asymptotically stable and the region of attraction includes the set $\mathcal{R}$. ■

Corollary 1: Suppose in addition to assumptions (A1), (A2) the functions f and κ are Lipschitz on compact sets. Then, there exists a constant $\tau_d > 0$ such that for all solutions starting in $\mathcal{R} \setminus \mathcal{A}$ the jumps are separated by at least an amount of time τ_d.

Proof: Outside the set $\mathcal{A}$, $x_q^+ \neq x_q$ and x_q^+ is given by the second case of (23). Further, (22) implies that after a jump $x \in \overline{C_\rho^k}$, where k is such that $x_q^+ = \omega^k$. Therefore, $|x_q^+ - x|/(W_R|x|) \leq \rho < 1$. The rest of the proof follows from an analysis similar to that in [2]. ■

In Theorem 1, the set $\mathcal{A}$ is globally asymptotically stable if W_R is a global constant. Notice that in event-triggered control, the measurement error is reset to zero at triggering instants. However, in the proposed discrete-event controller

[2]See [16] for the definitions of the terms ‘forward invariance’, ‘maximal solution’ and ‘complete solution’.

$|x_q^+ - x| \neq 0$ and instead satisfies $|x_q^+ - x| \leq \rho W|x|$, which is not zero in general. This is the reason why hysteresis is required in the quantizer to avoid chattering.

Next, we demonstrate that a quantizer satisfying (18)-(23) indeed exists, and construct a minimal set of generating points Ω that satisfy (18)-(22). For the sake of clarity, we first outline the design process for $n = 1$, that is, for nonlinear systems (2) that are one dimensional.

A. Design of Ω in One Dimensional Systems

Now, we invert the problem and ask: given a point in the region of interest what are the values ω^k, (18), can take such that C^k, (19), contains that point. If the point is 0 then it is contained in C^0. Also, all cells other than C^0 are intervals. Therefore, we ask the more specific question: given $r_k^u \neq 0$ such that $|r_k^u| \leq R$, what should ω^k be such that $|\omega^k| \leq |r_k^u|$ and $|\omega^k - r_k^u| = \rho W_R |r_k^u|$, where $0 < \rho < 1$ is a constant. Thus, r_k^u is the upper or the outer extreme of the interval C_ρ^k (see Figure 1). Thus, ρ is a parameter that allows us to over-design. The inverse problem has the unique solution

$$\omega^k = (1 - \rho W_R) r_k^u \tag{24}$$

Then the inner or lower extreme of the interval C_ρ^k is

$$r_k^l = \frac{\omega^k}{1 + \rho W_R} \tag{25}$$

Therefore, the interval C_ρ^k is the open interval (r_k^l, r_k^u) or

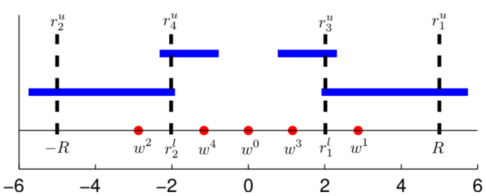


Fig. 1: Design of Ω for 1-D systems. The blue lines indicate the actual quantization cells or intervals, while r_k^u and r_k^l indicate the extremities of the over-designed quantization cells $\overline{C_\rho^k}$.

(r_k^u, r_k^l) depending on whether r_k^u is positive or negative, respectively. The points r_k^u and r_k^l are in the set C^k (see Figure 1). If we now set $r_{k+2}^u = r_k^l$ then we can recursively determine the set Ω. Following this procedure, we arrive at the following

$$\begin{aligned}
&\omega^0 = 0, \quad r_1^u = R, \quad r_2^u = -R \\
&\omega^k = (1 - \rho W_R) r_k^u, \quad \forall k \in \{1, 2, \ldots\} \\
&r_k^l = \frac{\omega^k}{1 + \rho W_R}, \quad \forall k \in \{1, 2, \ldots\} \\
&r_{k+2}^u = r_k^l, \quad \forall k \in \{1, 2, \ldots\} \\
&\omega^{k+2} = \frac{1 - \rho W_R}{1 + \rho W_R} \omega^k, \quad \forall k \in \{1, 2, \ldots\}
\end{aligned} \tag{26}$$

Note the symmetry in the positive and negative generators ω^k. Simple calculations give the quantization density as

$$\eta_q = \frac{2}{\ln\left(\frac{(1+\rho W_R)}{(1-\rho W_R)}\right)} \tag{27}$$

Thus, the proposed quantizer is a finite density logarithmic quantizer. The design process is summarized in Figure 1.

B. Design of Ω in Two Dimensional Systems

Due to space constraints, only a broad outline of the design process for the 2-D case is provided here. In the two-dimensional case, the design process is more complex due to the variety of possible cells, Figure 2a. Moreover, unlike in the 1-D case only a heuristically minimal solution is provided.

The algorithm progresses in stages by covering recursively one annulus after another with quantization cells. The process of determining these annuli is similar to the 1-D case, with the difference that the procedure (26) now gives the inner and outer radii (in the max norm sense) of the overlapping annuli, Figure 2b. Two completely covered overlapping annuli are shown in Figure 2c. The procedure to cover each annulus is identical and each annulus has the same number of cells. Thus, a finite density logarithmic quantizer is obtained, with a quantization density given by

$$\eta_q = 4\left\lceil \frac{1 + \rho W_R}{\rho W_R} \right\rceil \frac{1}{\ln\left(\frac{(1+\rho W_R)}{(1-\rho W_R)}\right)} \tag{28}$$

where $\lceil . \rceil$ denotes the greatest integer function.

V. EXAMPLE

In this section, the proposed emulation based controller is illustrated through an example. Consider the second order nonlinear system

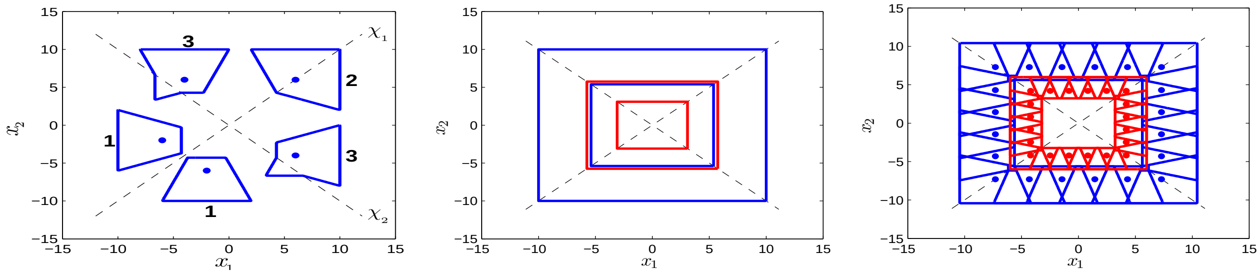
$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{l}(g \sin(x_1) + u).
\end{aligned} \tag{29}$$

Let the control input be given as

$$u = \kappa(x) = -l\lambda x_2 - g\sin(x_1) - K(x_2 + \lambda x_1). \tag{30}$$

where $K > 0$ and $\lambda > 0$. Let the Lyapunov function in assumption (A1) be $V(x) = \frac{l}{2}(x_2 + \lambda x_1)^2 + \lambda K x_1^2$. Routine calculations yield $W = 0.0447$ that satisfies the assumptions of Theorem 1, with the induced region of attraction being all of $\mathbb{R}^2$. Thus the discrete-event controller guarantees global asymptotic stability of the set $\mathcal{A}$ in the hybrid system $\mathcal{H}$, (6).

The quantizer designed as in Section IV with $\sigma = 0.99$ and $\rho = 0.9$ has a density ≈ 2582. Figure 3 shows the evolution of $|x|$ and $|e|/W$ for a sample trajectory. In the simulations the parameters g, l, K and λ were chosen as 10, 0.2, 1 and 1, respectively. The number of jumps or equivalently the number of control updates was observed to be 165 in the simulated time, giving an average update frequency of 33Hz. The minimum inter-update time was observed to be 0.0011s.



(a) The blue dots are the generators of the quantization cells, whose boundaries are represented by the blue polygons.

(b) The inner and outer boundaries of the first annulus are shown in blue, while those of the second annulus are shown in red.

(c) Quantization cells covering two overlapping annuli.

Fig. 2: Design of Ω for 2-D systems. Possible types of cells in the two-dimensional case (Figure 2a), the steps in designing the quantization cells (Figures 2b, 2c). The procedure leads to a finite density logarithmic quantizer.

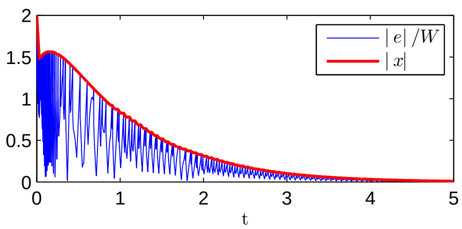


Fig. 3: Evolution of $|x|$ and $|e|/W$.

VI. Discussion and Conclusions

This paper revisits the problem of control under data-rate constraints. Specifically, we have combined the ideas of event-triggered control and coarsest quantization to propose a method for co-designing the event-trigger and the quantizer in emulation based controllers for stabilization tasks. The resulting quantizer is a finite density logarithmic quantizer, applicable to general multi-input and multi-dimensional continuous-time nonlinear systems. To the best of our knowledge, this work is the first to look at the co-design of the event-trigger and the quantizer in emulation based discrete-event controllers. The proposed design algorithm results in a controller that guarantees semi-global asymptotic stability of the origin of the system with a specified arbitrary compact region of attraction. In case a certain Lipschitz constant is global, the origin is globally asymptotically stable. If only semi-global practical stability is desired, with any specified compact region of attraction and ultimate bound, the quantizer has a finite number of cells.

Several extensions to the problem are possible, such as treating W itself as a state that is updated during the jumps. Such a formulation will lead to a more generalized quantizer design with lower densities. Finally, as mentioned in Section III, the proposed design easily extends to a case with a weaker assumption than the ISS one, the details of which will be communicated in future.

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