Event-Triggered Dynamic Output Feedback Control for LTI Systems

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Abstract— In this paper we propose a systematic methodology for designing implicitly verified event-triggered dynamic output feedback controllers for LTI systems that are observable and controllable. Event-triggering conditions that depend only on local information are proposed for sampled-data implementation of the observer and the controller in three different architectures. It is demonstrated that the triggering conditions provide a global lower bound on the inter-sample times and guarantee asymptotic stability of the closed loop system. The proposed design methodology is illustrated through simulation results.

I. INTRODUCTION

Recently, event-triggered control has been proposed as an alternative to the traditional time-triggered (example: periodic triggered) approach to digital control. In event based control systems, the timing of control execution is implicitly determined by a state dependent event-triggering condition. Thus, by encoding the nature of the task (for example stabilization) into an event-triggering condition, it is possible to systematically design controllers that make better use of computational and communication resources.

In recent years, event-triggered controllers have been systematically designed (a representative list includes [1], [2], [3], [4] and references therein) for different stabilization tasks. Most work in this literature assumes the availability of full state information. However, in many practical applications only a part of the state information can be directly measured and a dynamic (for example, observer based) output feedback controller must be utilized. Thus, it is important to develop event-triggered implementations of dynamic output feedback controllers and this paper contributes to the presently limited literature on the subject.

The main **contribution** of this paper is a methodology for designing implicitly verified event-triggered dynamic output feedback controllers for Linear Time Invariant (LTI) systems that are observable and controllable. We consider architectures where the sensor and the dynamic controller are co-located as well as one where they are not co-located. The proposed event-triggering conditions depend only on local information and include explicit positive lower thresholds for inter-sampling times that are designed to ensure global asymptotic stability of the closed loop system.

In comparison, [5] proposed an event-triggered implementation that guarantees uniform ultimate boundedness of the plant state and a semi-global estimate of the minimum inter-communication time (dependent on the initial state of the dynamic controller and the unknown state of the plant).

In [6], a model based output feedback controller was proposed, in which the communication from the observer subsystem to the system model subsystem is triggered whenever a condition comparing the observer state and the state of a local copy of the system model exceeds a threshold. This controller guarantees a positive minimum transmission time and uniform ultimate boundedness of the closed loop state. In [7], [8] event-triggered output feedback control for *discrete-time systems* was studied as an optimal control problem, explicitly involving communication costs. The proposed architecture includes a Kalman filter in the sensor subsystem and identical observers in the sensor as well as actuator subsystems. The results provide an upper bound on the optimal cost attained by the event-triggered system. In comparison to [6], [7], [8], we do not require identical observers/models to be run at different locations.

Recently, a self-triggered dynamic output feedback controller was proposed in [9] where a discrete-time observer is in cascade with a full state-feedback self-triggered controller. The resulting closed loop system is rendered input-to-state stable (ISS) with respect to exogenous disturbances. Finally, we have extended the proposed designs in this paper to decentralized LTI systems [10].

The rest of the paper is organized as follows. In Section II the problem under consideration is described and some mathematical notation is established. In Section III the dynamic controllers and the event-triggering conditions for different architectures are designed. This section also contains the main analytical results of this paper. The proposed design methodology is illustrated through simulations in Section IV and finally some concluding remarks are made in Section V.

II. PROBLEM STATEMENT

Consider a Linear Time Invariant (LTI) control system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \tag{1}$$

$$y = Cx, \quad y \in \mathbb{R}^p \tag{2}$$

Assume that (A, B) and (A, C) are controllable and observable, respectively. Further, assume that the gain matrices F and K (which exist) are such that $(A + FC)$ and $(A + BK)$ are Hurwitz. Thus, the continuous-time dynamic controller

$$\dot{\hat{x}} = (A + FC)\hat{x} + BK\hat{x} - Fy, \quad \hat{x} \in \mathbb{R}^n \tag{3}$$

$$u = K\hat{x} \tag{4}$$

is such that the closed loop system (1)-(4) is globally asymptotically stable. To see this, let us first denote the *observer estimation error* as

$$\tilde{x} \triangleq \hat{x} - x$$

Let $\psi \triangleq [x; \tilde{x}]$ be the aggregate state vector, where the notation $[a_1; a_2]$ denotes the column vector formed by concatenating the vectors a_1 and a_2. Then, the closed loop system may be written as

$$\dot{\psi} = \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A + BK & BK \\ 0 & A + FC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \triangleq \bar{A}\psi \tag{5}$$

where 0 represents a matrix of zeroes of appropriate size. There exist gains F and K such that the matrix $\bar{A}$ is Hurwitz, thus rendering the closed loop system stable.

In this paper, we are interested in event-triggered implementation of the dynamic controller (3)-(4). In sampled-data implementations (of which event-triggered implementation is an example) the data used by the controller and actuator are sampled at discrete time instants. Thus, let z be any continuous-time signal and let $\{t_i^z\}$ be the increasing sequence of time instants at which z is sampled. Then we denote the resulting piecewise constant sampled signal by z_s, that is,

$$z_s \triangleq z(t_i^z), \quad \forall t \in [t_i^z, t_{i+1}^z) \tag{6}$$

The sampled data, z_s, may also be viewed as resulting from an error in the the measurement of the continuous-time signal, z. This measurement error is denoted by

$$z_e \triangleq z_s - z = z(t_i^z) - z, \quad \forall t \in [t_i^z, t_{i+1}^z)$$

In time-triggered implementations, the time instants t_i^z are pre-determined and are commonly a multiple of a fixed sampling period. However, in event-triggered implementations the time instants t_i^z are determined implicitly by a state/data based triggering condition at run-time. Consequently, an event-triggering condition may result in the inter sampling times $t_{i+1}^z - t_i^z$ to be arbitrarily close to zero or it may even result in the limit of the sequence $\{t_i^z\}$ to be a finite number (*Zeno* behavior). Thus for practical utility, an event-trigger has to ensure that these scenarios do not occur.

In this paper we consider three different architectures of the closed loop system, shown in Figure 1. In the first two architectures, Figures 1a and 1b, the sensor and the controller are assumed to be co-located. *Co-located* components have access to each others' outputs at all times. Thus in Architecture I, the sensor output is used by the controller without sampling. In Architecture II, the controller utilizes the sampled version of the sensor output even though it has access to the continuous-time signal y. The utility of this architecture is discussed in the sequel. In this architecture, a single event-triggering condition triggers the sampling of both the sensor, y, and the controller output, u. Thus these signals are sampled synchronously, that is, $t_i^y = t_i^u$ for each $i \in \{0, 1, \ldots\}$. In Architecture III, the controller and the sensor are not co-located. Hence, the sensor output, y, and the controller output, u, are sampled asynchronously

($t_i^y \neq t_i^u$) by separate event-triggers, each depending only on local information.

The problem we are interested in this paper is the design of the dynamic controller and the event-triggers for each of the architectures shown in Figure 1. The resulting event-triggered controllers must render the closed loop system globally asymptotically stable and ensure that the inter-sampling times of each signal is lower bounded by a positive constant.

Finally, a point regarding the notation in the paper is that depending on the context, the notation $\|.\|$ denotes either the Euclidean norm of a vector or the induced matrix Euclidean norm. In the next section, the dynamic controllers and the event-triggering conditions for the three architectures are developed.

III. EVENT-TRIGGERED IMPLEMENTATIONS OF THE DYNAMIC CONTROLLER

In this section, the dynamic controllers and the event-triggering conditions are developed for the three architectures shown in Figure 1.

A. Architecture I

In Architecture I, the observer and the sensor are co-located, which means the observer has access to the sensor's output at all times. The closed loop system with the sampled data implementation of the observer and the controller is given by

$$\dot{x} = Ax + Bu_s \tag{7}$$
$$y = Cx \tag{8}$$
$$\dot{\hat{x}} = (A + FC)\hat{x} + BK\hat{x}_s - Fy \tag{9}$$
$$u = K\hat{x} \tag{10}$$

where the subscript s denotes the sampled versions of the corresponding continuous-time signals. The second term, $BK\hat{x}_s$, in the observer, (9), is the natural choice to model the effect of the sampled data control $u_s = K\hat{x}_s$ in the plant dynamics (7).

The closed loop system can be written in terms of the measurement error $\hat{x}_e = \hat{x}_s - \hat{x}$ as

$$\dot{\psi} = \bar{A}\psi + \begin{bmatrix} BK \\ 0 \end{bmatrix} \hat{x}_e = \bar{A}\psi + G_1 H_1 \psi_e \tag{11}$$

$$\text{where } G_1 \triangleq \begin{bmatrix} BK \\ 0 \end{bmatrix}, \quad H_1 \triangleq \begin{bmatrix} I_n & I_n \end{bmatrix} \tag{12}$$

where $\psi = [x; \tilde{x}] = [x; \hat{x} - x]$, $\bar{A}$ is as defined in (5), 0 represents a matrix of zeroes of appropriate size and I_n is the $n \times n$ identity matrix. Note that the sampled-data nature of the system is implicit in the measurement error term, $\hat{x}_e$ (or ψ_e). The system description is still incomplete and requires specification of the event-triggering condition.

To design the event-triggering condition we first present the following lemma, which is a modified version of Corollary IV.1 from [1].

Lemma 1: Let $\dot{\zeta} = \mathcal{A}\zeta + \mathcal{B}_1 \xi_e + \mathcal{B}_2 \omega_e$ denote a LTI control system where $\zeta \in \mathbb{R}^{n_\zeta}$, $\xi_e \in \mathbb{R}^{m_1}$, $\omega_e \in \mathbb{R}^{m_2}$, while $\mathcal{A}$,
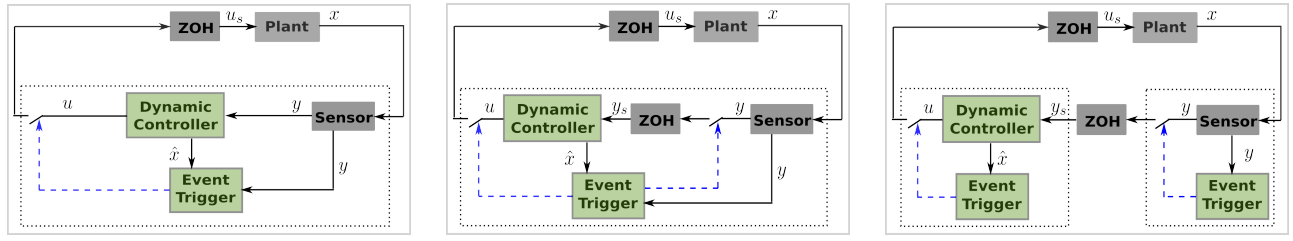
(a) Architecture I: Sensor and controller co-located. Sensor output available to the controller at all time.

(b) Architecture II: Sensor and controller co-located. Synchronous sampling of the sensor and controller outputs.

(c) Architecture III: Sensor and controller *not* co-located. Asynchronous sampling of the sensor and controller outputs.

Fig. 1: The three control loop architectures considered in this paper. The components within a dotted box are at the same location - that is each component has access to the other components' output at any given time.

$\mathcal{B}_1$ and $\mathcal{B}_2$ are matrices of appropriate dimensions. Further, $\xi_e \triangleq \xi_s - \xi$, $\xi = K_1\zeta$, $\omega_e \triangleq \omega_s - \omega$, $\omega = K_2\zeta$ where ξ_s and ω_s are defined as the piecewise constant signals obtained by asynchronously sampling ξ and ω, respectively, as in (6). Let $w_1 \geq 0$ and $w_2 \geq 0$ be any arbitrary constants. Suppose the sampling times are such that $\|\xi_e\|/\|\zeta\| \leq w_1$ and $\|\omega_e\|/\|\zeta\| \leq w_2$ for all time. Then, the time required for $\|\xi_e\|/\|\zeta\|$ to evolve from 0 to w_1 and time required for $\|\omega_e\|/\|\zeta\|$ to evolve from 0 to w_2 are lower bounded by τ_1 and τ_2, where

$$\tau_1 = \tau(w_1, \|\mathcal{A}\| + \|\mathcal{B}_2\|w_2, \|\mathcal{B}_1\|, \|K_1\|)$$
$$\tau_2 = \tau(w_2, \|\mathcal{A}\| + \|\mathcal{B}_1\|w_1, \|\mathcal{B}_2\|, \|K_2\|)$$

The function τ is given by

$$\tau(w, a, b, k) = \{t \geq 0 : \phi(t, 0) = w\} \tag{13}$$

where $w \geq 0$, a, b, k are positive constants and $\phi(t, c)$ is the solution of

$$\dot{\phi} = (k + \phi)(a + b\phi), \quad \phi(0, c) = c.$$

Proof: By direct calculation we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\xi_e\|}{\|\zeta\|}\right) = \frac{-(\xi_e^T\xi_e)^{-1/2}\xi_e^T K_1\dot{\zeta}}{\|\zeta\|} - \frac{\zeta^T\dot{\zeta}\|\xi_e\|}{\|\zeta\|^3}$$

$$\leq \left(\|K_1\| + \frac{\|\xi_e\|}{\|\zeta\|}\right)\frac{\|\dot{\zeta}\|}{\|\zeta\|}$$

$$\leq \left(\|K_1\| + \frac{\|\xi_e\|}{\|\zeta\|}\right)\frac{\|\mathcal{A}\|\|\zeta\| + \|\mathcal{B}_1\|\|\xi_e\| + \|\mathcal{B}_2\|\|\omega_e\|}{\|\zeta\|}$$

where for $\xi_e = 0$ the relation holds for all directional derivatives. Now, let $\nu_1 \triangleq \|\xi_e\|/\|\zeta\|$ and $\nu_2 \triangleq \|\omega_e\|/\|\zeta\|$. Then, it follows that

$$\dot{\nu}_1 \leq (\|K_1\| + \nu_1)(\|\mathcal{A}\| + \|\mathcal{B}_2\|w_2 + \|\mathcal{B}_1\|\nu_1)$$

where w_2 appears due to the assumption that the sampling instants are such that $\|\omega_e\|/\|\zeta\| \leq w_2$ for all time. An analogous expression for a bound on the rate of change of $\|\omega_e\|/\|\zeta\|$ can be obtained. The claim of the Lemma follows directly by comparison with (13). ∎

The following result presents the triggering condition (with an explicit positive lower bound on the inter-sampling times) for Architecture I and demonstrates global asymptotic stability of the closed loop system.

In the sequel the following notation is utilized.

$$\mathrm{Lyap}(A, Q) = P \text{ such that } PA + A^T P = -Q$$

where A is a square Hurwitz matrix and Q is a positive definite matrix.

Theorem 1: Consider the system given by (11) with (A, B) controllable, (A, C) observable and $\bar{A}$ Hurwitz. Let $Q \in \mathbb{R}^{2n}$ be any positive definite matrix and let the event-triggering condition be

$$t_{i+1} = \min\{t \geq t_i + T : \eta \geq 0\} \tag{14}$$

$$\eta = 2\|PG_1\|\|\hat{x}_e\| - \sigma Q_m\left(\frac{\theta_1\|\hat{x}\|}{\sqrt{2}} + \frac{\theta_2\|y\|}{\|C\|}\right) \tag{15}$$

where Q_m is the smallest eigenvalue of Q, $0 < \sigma < 1$, $0 \leq \theta_1 \leq 1$, $0 \leq \theta_2 \leq 1$ are design parameters such that $\theta_1 + \theta_2 = 1$, $P = \mathrm{Lyap}(\bar{A}, Q)$ is a positive definite matrix and $T = \tau\left(\frac{\sigma Q_m}{2\|PG_1\|}, \|\bar{A}\|, \|G_1\|, \|H_1\|\right)$, the function τ being as defined in (13). Then, the origin of the closed loop system is globally asymptotically stable and the inter-sample times are lower bounded by T.

Proof: First note that $\bar{A}$ is Hurwitz and hence there exists a positive definite matrix P that satisfies the Lyapunov condition in the statement of the theorem. Now consider the candidate Lyapunov function $V = \psi^T P\psi$, whose derivative along the flow of the closed loop system is given by

$$\dot{V} = \psi^T P\bar{A}\psi + \psi^T\bar{A}^T P\psi + 2\psi^T PG_1H_1\psi_e$$

$$\leq -\psi^T Q\psi + 2\psi^T PG_1\hat{x}_e$$

$$\leq -(1-\sigma)Q_m\|\psi\|^2 - \|\psi\|\left[\sigma Q_m\|\psi\| - 2\|PG_1\|\|\hat{x}_e\|\right] \tag{16}$$

Now, if $\hat{x} \neq 0$ then there is a non-negative number k such that $\|x\| = k\|\hat{x}\|$, from which it follows that

$$\|\psi\| = \sqrt{\|x\|^2 + \|\hat{x} - x\|^2} \geq \sqrt{\|x\|^2 + 2\|x\|^2 - 2\|x\|\|\hat{x}\|}$$

$$\geq \|\hat{x}\|\sqrt{2k^2 - 2k + 1} \geq \frac{\|\hat{x}\|}{\sqrt{2}}$$

where the last step follows by minimizing with respect to k. If on the other hand, $\hat{x} = 0$, then $\|\psi\| \geq \|\hat{x}\|/2$ holds trivially. Next, we see that $\|\psi\| \geq \|x\| \geq \|y\|/\|C\|$. Thus,

the following holds

$$\|\psi\| \geq \frac{\theta_1\|\hat{x}\|}{\sqrt{2}} + \frac{\theta_2\|y\|}{\|C\|}, \quad \text{with } \theta_1 + \theta_2 = 1$$

Hence the triggering condition (14) ensures that

$$\dot{V} \leq -(1-\sigma)Q_m\|\psi\|^2$$

from which it follows that the origin of the closed loop system is globally asymptotically stable. Next, from (16), it is clear that $\dot{V} \leq -(1-\sigma)Q_m\|\psi\|^2$ as long as

$$\frac{\|\hat{x}_e\|}{\|\psi\|} \leq w_1 = \frac{\sigma Q_m}{2\|PG_1\|} \tag{17}$$

Since $w_1 > 0$ and the norms of all the system matrices are finite, it follows from Lemma 1 (with $w_2 = 0$) that the inter-sample times are lower bounded by T, which is positive. ∎

Note that the time for the function η, (15), to grow to zero from its initial value at a sampling instant can be arbitrarily small. However, a lower bound on the time that the quantity on the left hand side of (17) takes to evolve from 0 to the threshold w_1 is independent of the unknown state of the closed loop system ψ and is given by a positive quantity, T.

In the next subsection, we design the dynamic controller and the event-trigger for Architecture II, Figure 1b.

B. Architecture II

In Architecture II, Figure 1b, the observer and the sensor are co-located, which means the observer has access to the sensor information at all times. However, the observer in this architecture utilizes sampled version of the sensor output. The controller and sensor outputs are sampled synchronously at time instants determined by a single event-trigger. The observer system in this case is given by

$$\dot{\hat{x}} = (A + FC)\hat{x} + BK\hat{x}_s - Fy_s \tag{18}$$

where the subscript s denotes the sampled versions of the corresponding continuous-time signals. The closed loop system may be written in terms of the measurement error $\hat{x}_e = \hat{x}_s - \hat{x}$ and $y_e = y_s - y$ as

$$\dot{\psi} = \bar{A}\psi + G_1\hat{x}_e + G_2y_e = \bar{A}\psi + GH\psi_e \tag{19}$$

$$G_2 \triangleq \begin{bmatrix} 0 \\ -F \end{bmatrix}, \quad G \triangleq \begin{bmatrix} BK & 0 \\ 0 & -F \end{bmatrix}, \quad H \triangleq \begin{bmatrix} I_n & I_n \\ C & 0 \end{bmatrix} \tag{20}$$

where $\psi = [x; \tilde{x}] = [x; \hat{x} - x]$, $\bar{A}$ is as defined in (5), G_1 is given by (12), 0 represents a matrix of zeroes of appropriate size and I_n is the $n \times n$ identity matrix. Note that the sampled-data nature of the system is implicit in the measurement error terms, $\hat{x}_e$ and y_e (or ψ_e). The utility of this architecture can be seen from (18). The observer in this architecture has piecewise constant inputs, which greatly simplifies the online computation of the observer state, $\hat{x}$.

The following result presents the triggering condition for Architecture II, which incorporates an explicit positive lower bound for the inter-sampling times, and demonstrates global asymptotic stability of the closed loop system.

Theorem 2: Consider the system given by (19) with (A, B) controllable, (A, C) observable and $\bar{A}$ Hurwitz. Let $Q \in \mathbb{R}^{2n}$ be any positive definite matrix and let the event-triggering condition be

$$t_{i+1} = \min\{t \geq t_i + T : \eta \geq 0\} \tag{21}$$

$$\eta = 2\|PG_1\|\|\hat{x}_e\| + 2\|PG_2\|\|y_e\|$$

$$\quad - \sigma Q_m\left(\frac{\theta_1\|\hat{x}\|}{\sqrt{2}} + \frac{\theta_2\|y\|}{\|C\|}\right) \tag{22}$$

where Q_m is the smallest eigenvalue of Q, $0 < \sigma < 1$, $0 \leq \theta_1 \leq 1$, $0 \leq \theta_2 \leq 1$ are design parameters such that $\theta_1 + \theta_2 = 1$, $P = \text{Lyap}(\bar{A}, Q)$ and $T = \tau\left(\frac{\sigma Q_m}{2\|PG\|}, \|\bar{A}\|, \|G\|, \|H\|\right)$, the function τ being as defined in (13). Then, the origin of the closed loop system is globally asymptotically stable and the inter-sample times are lower bounded by T.

The proof is similar to that of Theorem 1 and Lemma 1 is again invoked with $w_2 = 0$ because the controller and sensor outputs are sampled synchronously. In the next subsection, we design the dynamic controller and the event-triggers for Architecture III where the sampling is asynchronous.

C. Architecture III

In Architecture III, Figure 1c, the observer and the sensor are *not* co-located, which implies that the observer has access only to the sampled sensor data. Moreover, the controller and sensor outputs are sampled asynchronously at time instants determined by separate event-triggers. The observer in this case is given by

$$\dot{\hat{x}} = (A + FC)\hat{x} + BK\hat{x}_s - Fy_s \tag{23}$$

where the subscript s denotes the sampled versions of the corresponding continuous-time signals. It is to be understood implicitly that $\hat{x}_s$ and y_s represent asynchronously sampled signals, that is,

$$\hat{x}_s = \hat{x}(t_i^{\hat{x}}), \quad \forall t \in [t_i^{\hat{x}}, t_{i+1}^{\hat{x}})$$
$$y_s = y(t_i^y), \quad \forall t \in [t_i^y, t_{i+1}^y)$$

Then the closed loop system can be written in terms of the measurement errors, $\hat{x}_e = \hat{x}_s - \hat{x}$ and $y_e = y_s - y$ as

$$\dot{\psi} = \bar{A}\psi + G_1\hat{x}_e + G_2y_e \tag{24}$$

where $\psi = [x; \tilde{x}] = [x; \hat{x} - x]$, 0 represents a matrix of zeroes of appropriate size, $\bar{A}$, G_1 and G_2 are defined as in (5), (12) and (20), respectively. Note that

$$\hat{x} = \begin{bmatrix} I_n & I_n \end{bmatrix}\psi \triangleq H_1\psi, \quad y = \begin{bmatrix} C & 0 \end{bmatrix}\psi \triangleq H_2\psi \tag{25}$$

where I_n is $n \times n$ identity matrix. As in Architecture II, the observer has piecewise constant inputs, which simplifies the online computation of the observer state, $\hat{x}$. The final result, which is analogous to those for Architectures I and II is now presented.

Theorem 3: Consider the closed loop system given by (24) with (A, B) controllable, (A, C) observable and $\bar{A}$

Hurwitz. Let $Q \in \mathbb{R}^{2n}$ be any positive definite matrix and let the event-triggering condition be

$$t_{i+1}^{\hat{x}} = \min\{t \geq t_i^{\hat{x}} + T_1 : \eta_1 \geq 0\} \tag{26}$$

$$\eta_1 = 2\|PG_1\|\|\hat{x}_e\| - \frac{\sigma Q_m \theta_1}{\sqrt{2}}\|\hat{x}\| \tag{27}$$

$$t_{i+1}^{y} = \min\{t \geq t_i^{y} + T_2 : \eta_2 \geq 0\} \tag{28}$$

$$\eta_2 = 2\|PG_2\|\|y_e\| - \frac{\sigma Q_m \theta_2}{\|C\|}\|y\| \tag{29}$$

where Q_m is the smallest eigenvalue of Q, $0 < \sigma < 1$, $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ are design parameters such that $\theta_1 + \theta_2 = 1$ and $P = \mathrm{Lyap}(\bar{A}, Q)$. The inter-sampling time thresholds are given by

$$T_1 = \tau(w_1, \|\bar{A}\| + \|G_2\|w_2, \|G_1\|, \|H_1\|)$$
$$T_2 = \tau(w_2, \|\bar{A}\| + \|G_1\|w_1, \|G_2\|, \|H_2\|)$$

with the function τ being defined as in (13), H_1 and H_2 as defined in (25) while

$$w_1 = \frac{\sigma Q_m \theta_1}{2\|PG_1\|}, \quad w_2 = \frac{\sigma Q_m \theta_2}{2\|PG_2\|}$$

Then, the origin of the closed loop system is globally asymptotically stable and the inter-sample times of $\hat{x}$ and y are lower bounded by T_1 and T_2, respectively.

Proof: Consider the candidate Lyapunov function $V = \psi^T P \psi$, whose derivative along the flow of the closed loop system is given by

$$\begin{aligned}
\dot{V} &= \psi^T P \bar{A} \psi + \psi^T \bar{A}^T P \psi + 2\psi^T PG_1 \hat{x}_e + 2\psi^T PG_2 y_e \\
&\leq -\psi^T Q \psi + 2\psi^T PG_1 \hat{x}_e + 2\psi^T PG_2 y_e \\
&\leq -(1-\sigma)Q_m\|\psi\|^2 \\
&\quad - \|\psi\|\Big[\sigma Q_m \theta_1 \|\psi\| - 2\|PG_1\|\|\hat{x}_e\|\Big] \\
&\quad - \|\psi\|\Big[\sigma Q_m \theta_2 \|\psi\| - 2\|PG_2\|\|y_e\|\Big]
\end{aligned} \tag{30}$$

which implies

$$\begin{aligned}
\dot{V} &\leq -(1-\sigma)Q_m\|\psi\|^2 \\
&\quad - \|\psi\|\Big[\frac{\sigma Q_m \theta_1}{\sqrt{2}}\|\hat{x}\| - 2\|PG_1\|\|\hat{x}_e\|\Big] \\
&\quad - \|\psi\|\Big[\frac{\sigma Q_m \theta_2}{\|C\|}\|y\| - 2\|PG_2\|\|y_e\|\Big]
\end{aligned}$$

The triggering conditions then ensure that

$$\dot{V} \leq -(1-\sigma)Q_m\|\psi\|^2 \tag{31}$$

from which it follows that the origin of the closed loop system is globally asymptotically stable. Moreover, it is clear from (30) that (31) holds as long as

$$\frac{\|\hat{x}_e\|}{\|\psi\|} \leq w_1 = \frac{\sigma Q_m \theta_1}{2\|PG_1\|}, \quad \frac{\|y_e\|}{\|\psi\|} \leq w_2 = \frac{\sigma Q_m \theta_2}{2\|PG_2\|}$$

Since both w_1 and w_2 are strictly positive while the norms of all the system matrices are finite, we conclude from Lemma 1 that the lower bounds for inter-sampling times of $\hat{x}$ and y, T_1 and T_2, respectively are strictly positive. ∎

The design of the event-triggering condition and the proof of the Theorem 3 are very similar to those developed earlier. A minimum inter-sampling time is enforced by design. Given positive thresholds w_1 and w_2, Lemma 1 provides a positive minimum sampling time that is independent of the unknown state of the closed loop system. By appropriately choosing the thresholds w_1 and w_2, the resulting event-trigger is guaranteed to globally asymptotically stabilize the closed loop system. The triggering conditions only ensure that the inter-sample times of $\hat{x}$ and y are lower bounded by positive constants. It must be noted that it is possible for the controller output and the sensor output to be sampled arbitrarily close to each other. Another minor difference of this architecture with respect to the first two is that θ_1 and θ_2 cannot take values 0 or 1 but only those between them.

In the next section, simulation results are presented to illustrate the proposed event-triggered controllers.

IV. SIMULATION RESULTS

In this section, the proposed event-triggered dynamic output feedback controllers are illustrated. The following are the system matrices and parameters chosen for the simulations.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -15 & -10 \end{bmatrix}, \quad F = \begin{bmatrix} -10 \\ -14 \end{bmatrix}, \quad Q = I_4, \ \sigma = 0.95$$

where I_4 is the 4×4 identity matrix. For the simulations presented here, the initial condition of the plant and the observer were chosen as $x(0) = [2; 3]$ and $\hat{x}(0) = [0; 0]$, respectively. The zeroth sampling instant was chosen as $t_0^{\hat{x}} = -T$ in Architecture I and $t_0^{\hat{x}} = -T_1$ in Architectures II and III. Similarly, $t_0^{y} = -T_2$ was chosen in Architectures II and III. This is to ensure sampling at $t = 0$ if the $\psi_s(0)$ satisfies the triggering condition. In the simulations, $\hat{x}_s(0) = \hat{x}(0)$ was chosen in all the architectures. In Architecture II, $y_s(0) = y(0)$ was chosen because the observer and the sensor are co-located and the sampling is synchronous. In Architecture III, where the observer and the sensor are not co-located and whose outputs are sampled asynchronously, $\hat{x}_s(0) = \hat{x}(0)$ was chosen while $y_s(0) \neq y(0)$ was chosen as an arbitrary value. In the presented simulations (Sim Nos. 5 and 6) $y_s(0) = 1.01 \times y(0)$ was chosen.

Table I summarizes the results for the three architectures. We see that the triggering conditions ensure a lower bound on the inter-sampling times of the sensor and the controller outputs. The ratio of T_1 over the observed average sampling interval of $\hat{x}$ is given by μ_1. The variable μ_2 is similarly defined for y. Thus, μ_1 and μ_2 provide an indication of the communication resources required by the event-triggered sampling as a factor of time-triggered sampling, with a constant sampling period of T_1 and T_2, respectively. The lower the values of μ_1 and μ_2, the better it is.

Figure 2 shows the evolution of the Lyapunov function V in Sim Nos. 1, 3 and 5. As can be seen, the rate of convergence is only marginally faster in Sim No. 1 and is similar in all the three cases. Figures 3a and 3b show the

TABLE I: Simulation results for Architectures I, II and III. In Architecture I, the sensor output y is available to the controller all the time. In Architecture II, the controller and the sensor outputs are sampled synchronously by a single event-trigger.

	Sim. No.	θ_1	θ_2	T_1 (ms)	T_2 (ms)	Observed min. interval ($\hat{x}$) (ms)	Observed min. interval (y) (ms)	Observed Avg. interval ($\hat{x}$) (ms)	Observed Avg. interval (y) (ms)	μ_1	μ_2
Arch. I	1	1	0	2	-	2	-	7.9	-	0.25	-
Arch. I	2	0.6	0.4	2	-	2	-	7.1	-	0.28	-
Arch. II	3	1	0	0.57	0.57	0.57	0.57	3.8	3.8	0.15	0.15
Arch. II	4	0.6	0.4	0.57	0.57	0.85	0.85	3.4	3.4	0.17	0.17
Arch. III	5	0.5	0.5	1	0.48	1	0.48	4	1.6	0.25	0.31
Arch. III	6	0.3	0.7	0.61	0.67	0.61	0.67	2.5	2.2	0.25	0.31

evolution of the inter-sample times and the cumulative frequency distribution of the inter-sample times, respectively, in Sim Nos. 1, 3 and 5. As expected, Architecture I outperforms the other two in terms of the inter-sample times.
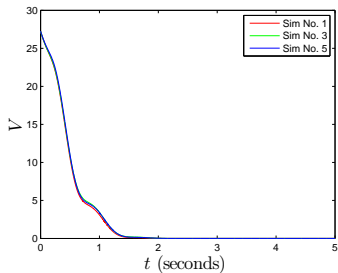

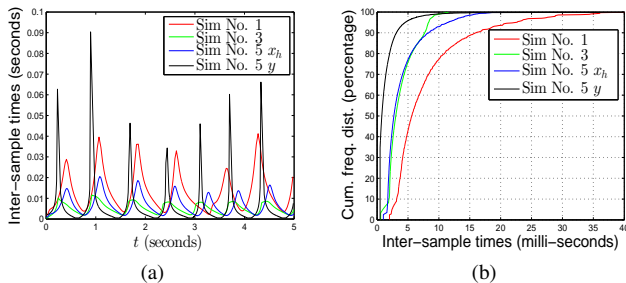Fig. 2: The evolution of the Lyapunov function V.

Fig. 3: (a) Evolution of the inter-sample times. (b) Cumulative frequency distribution of the inter-sample times.

Finally, simulations were also performed on the time-triggered architectures analogous to Architectures I, II and III. In each case, the constant sampling period was chosen as the average sampling periods shown in Table I. The convergence rates in these simulations were very similar to those shown above.

V. CONCLUSIONS

In this paper event-triggered dynamic output feedback controllers have been developed for architectures where the controller and the sensor are co-located as well as where they are not co-located. In each case, a minimum inter-sampling time is enforced by incorporating a lower threshold on inter-sampling interval in the event-triggering conditions.

The designed event-triggering conditions have been shown to ensure global asymptotic stability of the origin of the closed loop system. The proposed controllers have been illustrated through simulations.

The observer and the controller gains can be chosen independently, as in the classical case. However, their effect on the exact convergence rate and inter-sampling times has to be studied in detail. The inter-sample time thresholds can also be made less conservative. In Architecture III, the sensor and the controller are not co-located. Hence, the event-triggering conditions have been designed to sample the sensor and controller outputs asynchronously. This design can be extended to decentralized event-triggered implementations of dynamic output feedback controllers, which has been recently accomplished in [10]. Finally, although the proposed controllers represent an improvement over existing event-triggered dynamic output feedback controllers, simulation based comparison with time-triggered controllers suggests the inter-sample times are conservative.

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