Dynamics of a Stratified Population of Optimum Seeking Agents on a Network - Part I : Modeling and Convergence Analysis

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***Abstract*— We consider a population composed of a continuum of agents that seek to maximize a payoff function by moving on a network. The nodes in the network may represent physical locations or abstract choices. The population is stratified, i.e., agents opting for the same choice may not get the same payoff. In particular, we assume payoff functions with diminishing returns, i.e., agents in "newer" strata of a node receive a smaller payoff compared to "older" strata. In this first part of two-part work, we model the population dynamics under three choice revision policies, having varying levels of coordination — i. no coordination and the agents are selfish, ii. coordination among agents in each node and iii. coordination across the entire population. For the case with selfish agents, we generalize the Smith dynamics to our setting, where we have a stratified population and network constraints. To model nodal coordination, we allow the fraction of population in a node, as a whole, to take the 'best response' to the state of the population in the node's neighborhood. For the case of population-wide coordination, we explore a dynamics where the population evolves according to centralized gradient ascent of the social utility, though constrained by the network. In each case, we show that the dynamics has existence and uniqueness of solutions and also show that the solutions from any initial condition asymptotically converge to the set of Nash equilibria.**

***Index Terms*— Multi-agent systems, population dynamics, stratified population, Smith dynamics, best response dynamics, collective behavior, evolution on networks.**

I. Introduction

In large scale multi-agent systems, such as in population games and evolutionary dynamics, and swarm control, the evolution of the population as a whole is of greater interest than that of specific, individual agents. In this two-part work, we model and analyze the evolution of a population of optimum seeking and myopic agents that move on a network, under different levels of coordination, to maximize a payoff function.

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A. Literature Survey

A well established framework that explores large scale systems composed of strategic optimum seeking agents is that of population games and evolutionary dynamics [1]. This framework also finds application in distributed control and formation control problems [2]–[4]. However, in much of this literature, state dependent restriction on the actions available to the agents is missing. A natural way to model such restrictions is by having the nodes and edges of a network represent choices and the allowed transitions between the choices, respectively. There is considerable literature on evolutionary dynamics of finitely many agents on graphs, see for example [5]–[12], wherein the nodes and the edges of the graph represent the agents and the inter-agent interactions, respectively. Other works of literature [3], [13]–[15] let the nodes of the graph represent choices, with the state of the population being the fractions of population choosing each node. These references do consider the underlying graph in full generality but assume that the initial condition and the Nash Equilibrium can only be in the relative interior of the n-dimensional probability simplex. In [16] the continuum of replicators in each node of a graph is abstracted into a single player and the resulting dynamics is then explored. References [17], [18] explore imitation dynamics on a graph, where the nodes of the graph represent different communities and the edges represent the interaction between these communities. In all such works of literature, all the agents in a particular node receive the same payoff.

Another related area is that of swarms and swarm control. Reference [19] addresses control related problems for swarms, modeled as a continuum evolving in a continuous space, using partial differential equations. Markov chains [20]–[22] are also popular among works that consider probabilistic movement of agents from one cell to another in a discretized space. Reference [21] uses the convergence properties of Markov chains to design appropriate control actions for the swarm to reach a desired distribution. Reference [20] on the other hand works on designing the Markov chains so that the swarm converges to the stationary distribution.

B. Contributions

In this first part of our work, we study the evolution of a population of myopic agents that seek to maximize a payoff function by moving on a network. We model the population as a continuum of agents, with different fractions of the population located on different nodes in the network. The nodes in the network may represent physical locations or choices, in a more abstract sense, available to the infinitesimal agents. Unlike most works in literature, we consider a stratified population where agents choosing the same choice get different payoffs based on the strata they occupy. We study three different levels of coordination among the agents with an inherent desire to increase their payoff. At each time instant, the network imposes constraints on the set of choices that an agent can revise to. We model the evolution of the population on the network starting from an arbitrary initial state. Specifically, we characterize the dynamics, describe the set of Nash equilibria and demonstrate analytically that for all initial conditions, the trajectories converge to this set. Compared to [3], [13]–[15], in this work, we allow both the initial condition of the population and the equilibrium of the dynamics to be present anywhere on the n-dimensional probability simplex. Further, in our setup the set of Nash equilibria need not be a singleton and depends heavily on the graph structure. These make our model richer and the analysis significantly more challenging. Our work also features a non-linear dynamics with agents that actively seek the optimum. This differs from the literature on probabilistic swarm guidance [21], [23] where the transitions are often described by linear Markov chains. Compared to our preliminary work [24], here we consider arbitrary cumulative payoff functions that are strictly concave as opposed to quadratic functions and introduce a new dynamics called the stratified smith dynamics (SSD) that models selfish behavior. The more general payoff functions also make the analysis significantly more challenging and we thoroughly address the problem here.

In part two of this work [25], we present steady state analysis of a general class of dynamics that includes the dynamics proposed in this paper. In particular, we provide sufficient conditions on the graph under which these dynamics have a unique equilibrium point and in the case of a general graph, provide a computationally efficient method to compute bounds on the steady state value of the social utility.

C. Notation and Definitions

We denote the set of real numbers, the set of non-negative real numbers, and the set of integers using $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{Z}$, respectively. We $[p,q]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, p \leq x \leq q\}$. $\mathbb{R}^n$ (similarly $\mathbb{R}^n_+$) is the cartesian product of $\mathbb{R}$ (equivalently $\mathbb{R}_+$) with itself n times. If $\mathbf{v}$ is a vector in $\mathbb{R}^n$, we denote $\mathbf{v}_i$ as the i^{th} component of $\mathbf{v}$ and for a vector $\mathbf{v} \in \mathbb{R}^n$, we let $\text{supp}(\mathbf{v}) := \{i \in [1,n]_{\mathbb{Z}} \,|\, \mathbf{v}_i \neq 0\}$. A closed neighborhood around $\mathbf{v}$ with respect to the l-norm is denoted by $\mathcal{B}_l(\mathbf{v},r) := \{\mathbf{w} \in \mathbb{R}^n \,|\, \|\mathbf{w}-\mathbf{v}\|_l \leq r\}$. We let $\mathbf{1}$ be the vector, of appropriate size, with all its elements equal to 1 and we let $\mathbf{e}_i$ be the vector, again of appropriate size, with its i^{th} element equal to 1 and 0 for all other elements. The empty set is denoted by $\varnothing$. For two sets $\mathcal{U},\mathcal{V} \subset \mathcal{Q}$, the set subtraction operation is denoted by $\mathcal{U} \setminus \mathcal{V} = \mathcal{U} \cap \mathcal{V}^c$, where $\mathcal{V}^c$ is the set complement of $\mathcal{V}$ in $\mathcal{Q}$. If $\mathcal{Q}$ is an ordered countable set, then $\mathcal{Q}_i$ denotes the i^{th} member of $\mathcal{Q}$ and $|\mathcal{Q}|$ is used to represent the cardinality of $\mathcal{Q}$. $\{i,j\}$ is used to denote an unordered pair while (i,j) is used to denote an ordered pair. For a vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \geq 0$ is used to denote term wise inequalities. For a matrix $\mathbf{M} \in \mathbb{R}^{n\times m}$, the ij-th element of $\mathbf{M}$ is denoted by $[\mathbf{M}]_{ij}$. By $[.]_+ : \mathbb{R} \to \mathbb{R}_+$ we denote the function that is defined as $[x]_+ := \max\{x,0\}$. For a function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, ∇f is used to denote the gradient of $f(.)$ with respect to $\mathbf{x}$, *i.e.*, the j^{th} element of ∇f is $\frac{\partial f}{\partial \mathbf{x}_j}$.

II. Problem setup

In this paper, we consider a *population* composed of a continuum of *agents* that seek to maximize their *payoff* by moving on a network, with different levels of coordination. Let $\mathcal{V}$ be a set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ be a set of edges and $\mathcal{G} := (\mathcal{V},\mathcal{E})$ be an undirected graph that does not contain self loops or multiple edges between any pair of nodes. Let $N := |\mathcal{V}|$ be the total number of nodes and $M := |\mathcal{E}|$ be the total number of edges in the graph. The nodes in the network may either represent physical locations or *choices*, in a more abstract sense, that are available to the *infinitesimal agents* constituting the population. Let $\mathbf{x}_i \in [0,1]$ be the *fraction* of the population in node i, or equivalently making the choice i. We assume that the overall population is fixed and, without loss of generality, assume that $\sum_{i\in\mathcal{V}} \mathbf{x}_i = 1$.

We assume that the fraction in each node is stratified and the agents in different strata of a given node receive different payoffs. Let $p_i(.) : [0,1] \to \mathbb{R}$ be the function that models the *cumulative payoff* of the fraction $\mathbf{x}_i$. Let $[a,b] \subseteq [0,\mathbf{x}_i]$ be an arbitrary interval. Then the agents of node i that are in the *strata* $[a,b]$ get an *average payoff* of

$$\frac{p_i(b)-p_i(a)}{b-a}.$$

Thus the total or cumulative payoff that agents in node i receive is $p_i(\mathbf{x}_i)-p_i(0)$. Notice that for a node i if $a \in [0,\mathbf{x}_i]$,

$$u_i(a) := \frac{\mathrm{d}\,p_i}{\mathrm{d}\,y}(a)\,,$$

is the rate of change of the cumulative payoff at a and is also the average payoff that the agents in the strata $[a]$ of node i receive. By *strata* $[a]$ we mean the *infinitesimal strata* around a in $[0,\mathbf{x}_i]$. The agents in strata $[a]$ are aware of the payoff $u_i(a)$ that they receive $\forall a \in [0,\mathbf{x}_i], \forall i \in \mathcal{V}$. We call $u_i(.)$ as the *payoff density function* of node i. We let $u_i(0)$ be the right derivative of $p_i(.)$ at *zero* and $u_i(1)$ be the left derivative of $p_i(.)$ at *one*. We let $u(.)$ be the vector whose i^{th} element is $u_i(.)$. Through out this paper, we make the following assumption.

(A1) For all $i \in \mathcal{V}$, $p_i(.)$ is twice continuously differentiable and strictly concave. Hence, $\forall i \in \mathcal{V}$, $u_i(.)$ is a strictly decreasing function.

The function,

$$U(\mathbf{x}) := \sum_{i\in\mathcal{V}} \left[p_i(\mathbf{x}_i)-p_i(0)\right], \tag{1}$$

which we call as the *social utility function* is defined as the sum of the cumulative payoffs of agents in all the nodes. This represents the aggregate payoff that the population receives as a whole. Note that $U(.)$ is a strictly concave function and $\mathbf{e}_i^T \nabla U(\mathbf{x}) = u_i(\mathbf{x}_i), \forall \mathbf{x}$.

Let $\mathcal{N}^i$ be the set of all neighbors of node i in the graph $\mathcal{G}$ and let $\overline{\mathcal{N}}^i = \mathcal{N}^i \cup \{i\}$. Given the undirected graph $\mathcal{G}$, let

$$\mathcal{A} := \bigcup_{\{i,j\} \in \mathcal{E}} \{(i,j),(j,i)\}.$$

It is easy to see that $|\mathcal{A}| = 2|\mathcal{E}| = 2M$. The arcs in $\mathcal{A}$ are useful for denoting the inflow and outflow of the population fractions between adjacent nodes. Given a configuration $\mathbf{x}$, each infinitesimal agent in a node i may be able to increase its payoff by moving to its neighboring nodes $\mathcal{N}^i$. When an infinitesimal agent in node $i \in \mathcal{V}$ decides to switch to a node $j \in \mathcal{N}^i$, it gets to enter the *newest strata* $[\mathbf{x}_j]$ of node j. We impose this restriction in order to model diminishing returns, *i.e.* newer agents get lower payoff than older ones. Under this framework, we are interested in how the population as a whole evolves under different levels of coordination.

Remark 2.1: (Diminishing returns). As $u_i(.)$'s are strictly decreasing functions, the agents in a *newer strata* ($[a]$ for higher a) get lower payoffs than the ones in an *older strata* ($[a]$ for a lower a). Since we restrict the incoming agents to enter a node at the newest strata, they always get a payoff lesser than the older agents already residing in the node. •

Next, we briefly discuss the major features of the model proposed in this paper.

Remark 2.2: (Features of the model). One of the main features of our model is the graph that models the connections between choices. While this feature is not completely novel, it is quite rare in the literature on population dynamics. Such an interconnected network of choices naturally appears in problems where the choices are spatial locations. Examples include migrations of insect and animal swarms, control of robotic swarms and fleet redistribution in ride-sharing services. There are also scenarios where the inter-connected choices are more abstract, such as opinion dynamics of a population on inter-connected topics; and the inter-dependence of skills, credentials and jobs/professions in an economy.

The other major feature of the model is stratification, i.e., agents that subscribe to the same choice need not get the same payoff. Specifically in this work, we assume a diminishing returns model (see Remark 2.1) in that agents in a newer strata in a node receive a lower payoff. Such stratification is quite common in the financial and the economic world, such as investment in a stock or in real estate in an area. Quite often early investors get higher rate of returns. Similarly, early adopters of new disruptive technologies and skills have a significant edge in the job market, or in the case of businesses early adopters have a significant edge in capturing the market. Finally, again consider the example of a taxi or ride-sharing service. As the number of taxis in an area increases, the returns to the taxi drivers diminishes since the demand is fixed [26].•

In this paper, we consider the class of choice revision dynamics that can be expressed in the form

$$\dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}^i} [\delta_{ji}(\mathbf{x}) - \delta_{ij}(\mathbf{x})], \ \forall i \in \mathcal{V}, \tag{2}$$

where $\delta_{ij}(\mathbf{x}) \geq 0$ denotes the *outflow* of the fraction of population that moves from node i to node $j \in \mathcal{N}^i$ through the arc (i,j) as a function of the population state $\mathbf{x}$. We additionally impose the condition that $\delta_{ij}(\mathbf{x}) = 0 \ \forall j \in \mathcal{N}^i$ if $\mathbf{x}_i = 0$ in order to account for the fact that there cannot be any instantaneous outflow of population from a node if the node is empty. We call such dynamics as *flow balanced dynamics*. By choosing different sets of functions $\delta_{ij}(.)$'s, we can model different dynamics using (2). In this paper, we consider three specific dynamics with varying degrees of coordination among the agents as listed below.

- *Stratified Smith dynamics (SSD):* In the first dynamics, we assume that each agent is selfish and revises its choice at independent and random time instants. We model the evolution of the population's choice configuration by extending the standard Smith dynamics [1] to the case of stratified population. In this dynamics, whenever an agent gets an opportunity to revise its choice, it does so by comparing its payoff with the payoff of the agents at the newest strata in a neighboring node. We model and explore this dynamics in Section IV.
- *Nodal best response dynamics (NBRD):* In Section V, we assume that the agents coordinate with each other at the nodal level. In this case, the dynamics is the result of the fraction in each node i redistributing according to the best response of the fraction $\mathbf{x}_i$, as a whole, to the current configuration $\mathbf{x}$ while assuming that the fractions in the neighboring nodes do not change.
- *Network restricted payoff maximization (NRPM):* In Section VI, we assume that the agents coordinate across the entire population in a centralized manner. The population evolves according to network restricted gradient ascent of the social utility of the entire population.

For each dynamics, we analyze properties such as existence and uniqueness of solutions and convergence. In order to reduce repetition of ideas and analysis, we first discuss the general flow balanced dynamics in the following section.

III. Flow Balanced Dynamics

In this section, we discuss the general flow balanced dynamics, which can be described in terms of *inflows* and outflows of population between neighboring nodes. We provide sufficient conditions on a general flow balanced dynamics for existence and uniqueness of solutions. Then for a sub-class of flow balanced dynamics, we also discuss convergence of solutions.

Recall the flow balanced dynamics in (2) and recall that $\delta_{ij}(\mathbf{x})$ denotes the outflow from node $i \in \mathcal{V}$ to node $j \in \mathcal{N}^i$ as a function of the current state $\mathbf{x}$. Thus, the rate of change of $\mathbf{x}_i$ is given by the inflows from neighboring nodes to i minus the outflows from node i to its neighbors. In the sequel, we omit the argument of the outflow and denote it by δ_{ij} wherever there is no confusion.

For the directed graph $\mathcal{F} := (\mathcal{V}, \mathcal{A})$, let $\mathbf{J} \in \mathbb{R}^{N \times 2M}$ be the incidence matrix. In particular, we can number each arc in $\mathcal{A}$ and let $\mathcal{A}_m$ be the m^{th} arc, with $m \in [1 \ , \ 2M]_{\mathbb{Z}}$. If $\mathcal{A}_m = (i,j)$, then $[\mathbf{J}]_{im} = -1, [\mathbf{J}]_{jm} = 1$ and $[\mathbf{J}]_{km} = 0, \forall\, k \in \mathcal{V} \backslash \{i,j\}$. Similarly, we assemble the elements of the set $\{\delta_{ij}\}_{(i,j)\in\mathcal{A}}$ into the vector $\Delta \in \mathbb{R}^{2M}$ as

$$\Delta_m := \delta_{ij}, \text{ for } m \in [1\ ,\ 2M]_{\mathbb{Z}} \text{ s.t. } \mathcal{A}_m = (i,j). \quad (3)$$

Then, flow balanced dynamics (2) is concisely expressed as

$$\dot{\mathbf{x}} = \mathbf{J}\,\Delta(\mathbf{x}) =: \mathbf{F}(\mathbf{x})\,. \quad (4)$$

It is easy to see that the simplex,

$$\mathcal{S} := \{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{x} \geq 0\ ,\ \mathbf{1}^T\mathbf{x} = 1\},$$

which is a compact set, is positively invariant under (4). The following lemma, which we prove in Appendix D, gives a sufficient condition under which $\mathbf{F}(.)$ is locally Lipschitz and consequently (4) has existence and uniqueness of solutions for each initial condition in $\mathcal{S}$.

Lemma 3.1: (Existence and uniqueness of solutions for flow balanced dynamics). Suppose there exist closed sets $\{\mathcal{C}_i \subseteq \mathbb{R}^N_+\}_{i\in[1,n]_{\mathbb{Z}}}$ and functions $\{\mathbf{F}^i(.) : \mathbb{R}^N \to \mathbb{R}^N\}_{i\in[1,n]_{\mathbb{Z}}}$ such that $\bigcup_{i\in[1,n]_{\mathbb{Z}}} \mathcal{C}_i = \mathbb{R}^N_+$ and $\mathbf{F}(\mathbf{x})\big|_{\mathbf{x}\in\mathcal{C}_i} = \mathbf{F}^i(\mathbf{x})$, $\forall\, i \in [1,n]_{\mathbb{Z}}$, with $\mathbf{F}^i(\mathbf{x}) = \mathbf{F}^j(\mathbf{x}) = \mathbf{F}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C}_i \cap \mathcal{C}_j$. Suppose $\forall i \in [1,n]_{\mathbb{Z}}$, $\mathbf{F}^i(.)$ is locally Lipschitz in the domain $\mathcal{C}_i$. Then $\mathbf{F}(.)$ is Lipschitz on the domain $\mathcal{S}$. Further, for each $\mathbf{x}(0) \in \mathcal{S}$ system (4) has a unique solution $\forall\, t \geq 0$. •

Next we delve deeper into a subset of this general class of dynamics. The dynamics proposed in Sections IV and V fall into this subclass. Although the dynamics proposed in Section VI does not fall in this subclass, it still belongs to the general class of flow balanced dynamics. We later use results from this section to prove key results regarding the proposed dynamics.

A. Strongly Positively Correlated Dynamics

We call any dynamics that additionally satisfies the following criterion as *strongly positively correlated dynamics*.

Definition 3.2: (Strong positive correlation). We say $\mathbf{F}(.) : \mathbb{R}^N \to \mathbb{R}^N$ in (4) is strongly positively correlated with $u(.)$ if $u_i(\mathbf{x}_i) \geq u_j(\mathbf{x}_j)$ implies $\delta_{ij} = 0$, $\forall (i,j) \in \mathcal{A}$ and $\forall \mathbf{x} \in \mathcal{S}$. •

Definition 3.2 allows us to immediately analyze the convergence property of any strongly positively correlated dynamics. Note that the evolution of the state $\mathbf{x}$ is constrained by the network, and the dynamics (4) is closely determined by the incidence matrix. We first make a simple observation in the following lemma, which rules out cyclic outflows. We prove the lemma in Appendix A.

Lemma 3.3: (Strong positive correlation and acyclic flow). Consider the dynamics (4) and suppose $\mathbf{F}(.)$ satisfies strong positive correlation. For an arbitrary $\mathbf{x} \in \mathcal{S}$, suppose $\delta_{ij} \geq 0$, $\forall (i,j) \in \mathcal{A}$ and let $\tilde{\mathcal{A}} := \{(i,j) \in \mathcal{A} \,|\, \delta_{ij} > 0\}$. Then, the graph $\tilde{\mathcal{F}} := (\mathcal{V}, \tilde{\mathcal{A}})$ is a directed acyclic graph. •

Next, we characterize the equilibrium set of the dynamics (4). In general, there may be equilibrium points of the dynamics outside $\mathcal{S}$, but we are interested in the ones that are in the simplex $\mathcal{S}$. This equilibrium set is

$$\mathcal{X} := \{\mathbf{x} \in \mathcal{S} \mid \mathbf{F}(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathcal{S} \mid \Delta(\mathbf{x}) \in \ker(\mathbf{J})\}. \quad (5)$$

Given Lemma 3.3, we can further refine this set to the set of all $\mathbf{x} \in \mathcal{S}$ such that $\Delta(\mathbf{x}) = \mathbf{0}$. The following lemma characterizes an important subset of the set of equilibrium points of the dynamics, the significance of which is illustrated in the remark following the lemma. The proof of the result appears in Appendix A.

Lemma 3.4: (Non-emptiness of equilibrium set for strongly positively correlated dynamics). The unique optimizer of

$$\mathbf{P}_1 :\ \max_{\mathbf{x}} U(\mathbf{x}) = \max_{\mathbf{x}} \sum_{i\in\mathcal{V}} p_i(\mathbf{x}_i), \ \text{ s.t. }\ \mathbf{x} \in \mathcal{S}\,,$$

belongs to the set

$$\mathcal{NE} := \big\{\mathbf{x} \in \mathcal{S} \;\big|\; u_i(\mathbf{x}_i) \geq u_j(\mathbf{x}_j), \\ \forall\, j \in \mathcal{N}^i, \forall\, i \in \operatorname{supp}(\mathbf{x})\big\}\,. \quad (6)$$

Hence $\mathcal{NE}$ is non-empty. For the dynamics (4), suppose $\mathbf{F}(.)$ is strongly positively correlated with $u(.)$. Also let $\delta_{ij} = 0$ $\forall j \in \mathcal{N}^i$, if $\mathbf{x}_i = 0$. Then, $\mathcal{NE} \subseteq \mathcal{X}$, with $\mathcal{X}$ as in (5). •

Remark 3.5: (Nash equilibria). The set $\mathcal{NE}$ in (6) is the *Nash equilibria* of the population game. If the population configuration $\mathbf{x}$ is in $\mathcal{NE}$, then no agent has an incentive to unilaterally deviate from its node (or choice). Note that the revision choices available to an agent are dependent on its current choice. Moreover, notice that the set of Nash equilibria in (6) is the same as the one in the case where the population fractions are not stratified, that is where $u_i(.)$'s are the average payoff functions. Also note that the population configuration that globally maximizes the social utility always lies in $\mathcal{NE}$.•

Example 3.6: (Continuum of Nash equilibria). Consider a node set $\mathcal{V} = \{1,2,3\}$ and let the cumulative payoff functions be $p_i(y) = -0.5\,y^2 - a_i\,y$. Thus the payoff density functions are $u_i(y) = -y - a_i$. Let $a_1 = a_3 = 0$ and $a_2 = 5$. Consider the graph $\mathcal{G}^1$ with node set $\mathcal{V}$ and edge set $\mathcal{E}^1 = \{\{1,2\},\{2,3\}\}$. For this graph, the set of Nash Equilibria is $\mathcal{NE}^1 = \{\sigma[1,0,0]^T + (1-\sigma)[0,0,1]^T \,|\, \sigma \in [0,1]\}$. Now consider the graph $\mathcal{G}^2$ with the same node set $\mathcal{V}$ but edge set $\mathcal{E}^2 = \{\{1,2\},\{1,3\},\{2,3\}\}$. For this graph the set of Nash equilibria is $\mathcal{NE}^2 = \{[0.5,0,0.5]\}$ a singleton. Both graphs are connected but $\mathcal{NE}^2 \subset \mathcal{NE}^1$. This is because $\forall \overline{\mathbf{x}} \in \mathcal{NE}^1$, every path between nodes 1 and 3 (which are in $\operatorname{supp}(\overline{\mathbf{x}})$) has node $2 \notin \operatorname{supp}(\overline{\mathbf{x}})$. Thus the framework proposed in this paper is more general than the one considered in [14], [15]. •

Next, we give sufficient conditions under which the population evolving under any strongly positively correlated dynamics converges asymptotically to the set of equilibrium points.

Theorem 3.7: (Asymptotic convergence in strongly positively correlated dynamics). Consider the dynamics (4) with $\delta_{ij} \geq 0$ for all $(i,j) \in \mathcal{A}$ and for all $\mathbf{x} \in \mathcal{S}$. Also, suppose that $\mathbf{F}(.)$ is strongly positively correlated with $u(.)$. Then, $\mathcal{NE}$ is non-empty and $\mathcal{NE} \subseteq \mathcal{X}$, the set of equilibrium points of (4) in $\mathcal{S}$. Further, $\forall\, \mathbf{x}(0) \in \mathcal{S}$, $\mathbf{x}(t)$ asymptotically converges to $\mathcal{X}$ and $U(\mathbf{x}(t))$ converges to a constant.

Proof: Recall that Lemma 3.4 guarantees that $\mathcal{NE}$ is non-empty and that $\mathcal{NE} \subseteq \mathcal{X}$. In order to prove the claim about convergence, consider the Lyapunov-like function $V(.) := -U(.)$, see (1). Note that as $U(.)$ is strictly concave, $V(.)$ is strictly convex. Now, along the trajectories of (4),

$$\begin{aligned}\dot{V} &= (\nabla V)^T \mathbf{J}\,\Delta = -(\nabla U)^T \mathbf{J}\,\Delta \\ &= \sum_{(i,j)\in\mathcal{A}} \delta_{ij}\left[u_i(\mathbf{x}_i) - u_j(\mathbf{x}_j)\right]. \end{aligned} \quad (7)$$

This expression is readily obtained from the equivalence of (4) and (2) and after regrouping the terms. As $\mathbf{F}(.)$ in (4) is strongly positively correlated with $u(.)$, we have $u_i(\mathbf{x}_i) \geq u_j(\mathbf{x}_j)$ implies $\delta_{ij} = 0$, which is equivalent to $\delta_{ij} > 0$ implies $u_i(\mathbf{x}_i) < u_j(\mathbf{x}_j)$. The contrapositive has $\delta_{ij} > 0$ rather than $\delta_{ij} \neq 0$ as, again by the assumption of the theorem, $\delta_{ij} \geq 0$. We thus have $\dot{V} \leq 0$ and since the simplex $\mathcal{S}$ is positively invariant, LaSalle's invariance principle [27] says that $\mathbf{x}$ asymptotically converges to the set $\mathcal{X}$ and $U(\mathbf{x}(t))$ converges to a constant. ∎

In this paper, we are interested more in the class of dynamics that converges to a population state in the set of Nash Equilibria. The following corollary provides a sufficient condition for that to happen, namely $\mathcal{X} = \mathcal{NE}$.

Corollary 3.8: Suppose the hypothesis in Theorem 3.7 holds. In addition suppose $\mathbf{F}(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \notin \mathcal{NE}$. Then $\forall\, \mathbf{x}(0) \in \mathcal{S}$, $\mathbf{x}(t)$ asymptotically converges to $\mathcal{NE}$. •

IV. Stratified Smith Dynamics

In this section, we assume that the agents in the population are selfish and revise their choices independently. The dynamics we present for this scenario shares the spirit of the standard Smith dynamics [1] but is different due to stratified population fractions. We first derive the Smith dynamics for the stratified population setting. Then, we show that the dynamics satisfies the strong positive correlation property in Definition 3.2, which then immediately allows us to conclude about the convergence properties of the dynamics.

Recall that the $u_i(.)$ functions are strictly decreasing. Thus, the agents that are in a newer strata receive lower payoffs than the ones that are in an older strata. Recall that, when an infinitesimal agent in node $i \in \mathcal{V}$ decides to switch to a node $j \in \mathcal{N}^i$, it gets to enter the newest strata $[\mathbf{x}_j]$ of node j. Thus, agents in the strata $[y]$ of node $i \in \mathcal{V}$ compare their payoff density, $u_i(y)$, with the payoff density, $u_j(\mathbf{x}_j)$, of agents in strata $[\mathbf{x}_j]$ of node $j \in \mathcal{N}^i$ while deciding whether to switch to node j. Here, we assume that the agents in each node $i \in \mathcal{V}$ have the information about $u_j(\mathbf{x}_j)$ for its neighboring nodes $j \in \mathcal{N}^i$. The net outflow from a node $i \in \mathcal{V}$ to a node $j \in \mathcal{N}^i$ is a result of the agents in each population strata $[y]$ of i taking a decision to switch to j with a probability $[u_j(\mathbf{x}_j) - u_i(y)]_+$ (normalized appropriately) as in the Smith dynamics. Thus, the net outflow from i to j (in the expected sense) is

$$\delta_{ij} = \int_0^{\mathbf{x}_i} [u_j(\mathbf{x}_j) - u_i(y)]_+ \,\mathrm{d}y = \int_{y_{ij}}^{\mathbf{x}_i} [u_j(\mathbf{x}_j) - u_i(y)] \,\mathrm{d}y, \tag{8}$$

where

$$y_{ij} := \begin{cases} 0, & \text{if } u_i^{-1}(u_j(\mathbf{x}_j)) < 0 \\ u_i^{-1}(u_j(\mathbf{x}_j)), & \text{if } 0 \leq u_i^{-1}(u_j(\mathbf{x}_j)) \leq \mathbf{x}_i \\ \mathbf{x}_i, & \text{if } u_i^{-1}(u_j(\mathbf{x}_j)) > \mathbf{x}_i\,. \end{cases} \tag{9}$$

Here, $[y_{ij},\ \mathbf{x}_i]$ is the strata that moves from node i to node j. Thus, it must be that $u_i(y_{ij}) \leq u_j(\mathbf{x}_j)$, which determines the base case in (9). The other two cases are for restricting y_{ij} to $[0,\ \mathbf{x}_i]$. In the second equality of (8), we have used the fact that $u_i(.)$ is a strictly decreasing function. Note that as $u_i(.)$ is a strictly decreasing continuously differentiable function, its inverse also exists. So,

$$\delta_{ij} = [u_j(\mathbf{x}_j)(\mathbf{x}_i - y_{ij}) - (p_i(\mathbf{x}_i) - p_i(y_{ij}))]_+\,. \tag{10}$$

From (10), it is clear that $\delta_{ij} \geq 0$, $\forall (i,j) \in \mathcal{A}$. Then the dynamics in (4) with δ_{ij} defined in (10) is what we call as the *stratified Smith dynamics* (SSD).

Remark 4.1: (Existence and uniqueness of solutions for SSD). Recall that $\forall i \in \mathcal{V}$, $p_i(.)$ is a twice continuously differentiable function and hence $u_i(.)$ is strictly decreasing and continuously differentiable. As a result, the derivative of $u_i(.)$ is not zero at any point in [0,1]. Thus, by the inverse function theorem, $u_i^{-1}(.)$ is also differentiable and locally Lipschitz in [0,1]. Consequently, $\mathbf{J}\,\Delta(.)$ as a whole, is locally Lipschitz on $\mathcal{S}$. Then Lemma 3.1 guarantees the existence and uniqueness of solutions of SSD $\forall t \geq 0$ and for each initial condition $\mathbf{x}(0) \in \mathcal{S}$. •

In the next theorem, we first show that SSD satisfies Definition 3.2 and that the set of equilibrium points $\mathcal{X} = \mathcal{NE}$. Further, we also we show that all solutions of SSD converge asymptotically to the equilibrium set $\mathcal{X} = \mathcal{NE}$.

Theorem 4.2: (SSD: strong positive correlation and asymptotic convergence to the Nash equilibrium set). Suppose $\forall\, i \in \mathcal{V}$, $u_i(.)$ is strictly decreasing. Then, SSD, dynamics in (4) with δ_{ij} defined in (10), is strongly positively correlated with $u(.)$. The set of equilibrium points of SSD in $\mathcal{S}$, $\mathcal{X}$, is the set $\mathcal{NE}$ in (6). Further, $\forall \mathbf{x}(0) \in \mathcal{S}$, $\mathbf{x}(t)$ converges asymptotically to $\mathcal{NE}$ and $U(\mathbf{x}(t))$ converges to a constant.

Proof: Consider arbitrary nodes $i, j \in \mathcal{V}$. If $j \notin \mathcal{N}^i$ then by definition $\delta_{ij} = 0$. If $j \in \mathcal{N}^i$ and $\mathbf{x}$ is such that $u_i(\mathbf{x}_i) \geq u_j(\mathbf{x}_j)$ then as $u_i(.)$'s are strictly decreasing, we have $\mathbf{x}_i \leq u_i^{-1}(u_j(\mathbf{x}_j))$. Then, (9) implies $y_{ij} = \mathbf{x}_i$ and (8) further implies that $\delta_{ij} = 0$. Thus, SSD is strongly positively correlated with u.

Now, using Lemma 3.3, it is immediately evident that the only $\Delta(\mathbf{x}) \in \ker(\mathbf{J}) \cap \mathcal{S}$ is $\Delta(\mathbf{x}) = \mathbf{0}$. Thus $\mathbf{x}$ is an equilibrium point if and only if all $\delta_{ij}(\mathbf{x})$ are *zero*. If $i \notin \operatorname{supp}(\mathbf{x})$ then by (8) $\delta_{ij} = 0$, $\forall j \in \mathcal{N}^i$ is the only possibility. Now suppose that $i \in \operatorname{supp}(\mathbf{x})$. Observe from the definition of y_{ij} in (9) and (8) that $[u_j(\mathbf{x}_j) - u_i(y)] > 0$ for all $y \in [y_{ij},\ \mathbf{x}_i]$. Thus, the case $\delta_{ij} = 0$ and $\mathbf{x}_i > 0$ occur iff $y_{ij} = \mathbf{x}_i$, which according to (9) occurs iff $u_i(\mathbf{x}_i) \geq u_j(\mathbf{x}_j)$. Hence, the equilibrium set $\mathcal{X}$ of SSD, $\mathcal{X}$, is $\mathcal{NE}$ in (6).

The claim about asymptotic convergence of the trajectories to $\mathcal{NE}$ is a direct consequence of Theorem 3.7. ∎

V. Nodal Best Response Dynamics

In this section, we assume that the infinitesimal agents in each node $i \in \mathcal{V}$ coordinate and seek to maximize the overall payoff that the fraction $\mathbf{x}_i$ receives by redistributing itself among its neighbors. Thus, we call this dynamics as *nodal best response dynamics* (NBRD). In this section, we present the NBRD dynamics and analyze its properties such as existence and uniqueness of solutions and convergence.

Here, like in Section IV, the agents in a node react purely in response to the current population configuration and in fact assume that the fractions in other nodes would not revise their choices. The agents are aware of the payoff density functions and the population fractions of their node and their neighboring nodes. Further, if from a node $i \in \mathcal{V}$ a fraction of population $\delta_{ij} \leq \mathbf{x}_i$ decides to switch over to node $j \in \mathcal{N}^i$, they assume that they would occupy the strata $[\mathbf{x}_j, \mathbf{x}_j + \delta_{ij}]$ in node j. Under these assumptions, the fraction $\mathbf{x}_i$ in node i determines the reallocation $\{\mathbf{d}^*_{ij}\}_{j \in \overline{\mathcal{N}}^i}$ of $\mathbf{x}_i$ among the nodes i and its neighbors that maximizes the overall payoff of the fraction $\mathbf{x}_i$. Then, the outflows $\delta_{ij} = \mathbf{d}^*_{ij}$ for all $j \in \mathcal{N}^i$ and for all $i \in \mathcal{V}$. This behaviour is in the spirit of best response dynamics [1] in evolutionary dynamics but from the fraction $\mathbf{x}_i$'s point of view rather than from the point of view of infinitesimal agents.

Specifically, the outflows δ_{ij} from node i are equal to $\mathbf{d}^*_{ij}$, the optimizer of the problem

$$\begin{aligned} &\mathbf{P}^i_2: && (11)\\ &\max_{\{\mathbf{d}_{ij} | j \in \overline{\mathcal{N}}^i\}} \sum_{j \in \mathcal{N}^i} [p_j(\mathbf{x}_j + \mathbf{d}_{ij}) - p_j(\mathbf{x}_j)] + [p_i(\mathbf{d}_{ii}) - p_i(0)] \\ &\text{s.t. } \mathbf{d}_{ii} + \sum_{j \in \mathcal{N}^i} \mathbf{d}_{ij} = \mathbf{x}_i, \quad \mathbf{d}_{ij} \geq 0, \ \forall j \in \overline{\mathcal{N}}^i. \end{aligned}$$

Note that $p_j(\mathbf{x}_j + \mathbf{d}_{ij}) - p_j(\mathbf{x}_j)$ is the cumulative payoff received by the agents in strata $[\mathbf{x}_j, \mathbf{x}_j + \mathbf{d}_{ij}]$ of node $j \in \mathcal{N}^i$ if the fraction $\mathbf{d}_{ij}$ moves to node j and the agents in node j do not revise their choice. The non-negativity constraints $\mathbf{d}_{ij} \geq 0$ and $\mathbf{d}_{ii} \geq 0$ ensure that $\mathbf{d}_{ij}$'s are outflows and $\mathbf{d}_{ii}$ is the fraction remaining in node i. In order to model and analyze the dynamics of the population in such a scenario, we list some important properties of the optimizers of $\mathbf{P}^i_2$ in the following lemma. We provide its proof in Appendix B.

Lemma 5.1: (Optimizers of $\boldsymbol{P}^i_2$). For each $\mathbf{x} \in \mathcal{S}$, $\mathbf{P}^i_2$ in (11) possesses a unique optimizer, $\{\mathbf{d}^*_{ij}\}_{j \in \overline{\mathcal{N}}^i}$. Moreover, if $\mathbf{x}_i > 0$ then there exists a constant η_i such that

$$u_i(\mathbf{d}^*_{ii}) = \eta_i, \qquad \text{if } \mathbf{d}^*_{ii} > 0, \tag{12a}$$

$$u_j(\mathbf{x}_j + \mathbf{d}^*_{ij}) = \eta_i, \quad \forall j \in \mathcal{N}^i \text{ s.t. } \mathbf{d}^*_{ij} > 0, \tag{12b}$$

$$u_j(\mathbf{x}_j) \leq \eta_i, \qquad \forall j \in \overline{\mathcal{N}}^i \text{ s.t. } \mathbf{d}^*_{ij} = 0. \tag{12c}$$

•

Remark 5.2: (Nodal best response dynamics). $\mathbf{P}^i_2$ repeated for all $i \in \mathcal{V}$ gives the set $\{\mathbf{d}^*_{ij}\}_{(i,j) \in \mathcal{A}}$ of all outflows on every arc $(i,j) \in \mathcal{A}$, that is,

$$\delta_{ij} = \mathbf{d}^*_{ij}, \quad \forall (i,j) \in \mathcal{A}, \tag{13}$$

where $\{\mathbf{d}^*_{ij}\}_{(i,j) \in \mathcal{A}}$ is the subset of optimizers of $\{\mathbf{P}^i_2\}_{i \in \mathcal{V}}$. Then, the dynamics given by (4) with δ_{ij}'s as in (13) is what we refer to as nodal best response dynamics (NBRD). Note that if $\mathbf{x}$ is changed, the set of optimizers $\{\mathbf{d}^*_{ij}\}_{(i,j) \in \mathcal{A}}$ and hence Δ may change too. Thus $\Delta(\mathbf{x})$ is a function of $\mathbf{x}$. •

Application of Lemma 5.1 also lets us infer that NBRD is strongly positively correlated with $u(.)$, which we state in the following lemma, whose proof we provide in Appendix B.

Lemma 5.3: NBRD, dynamics (4) with $\delta_{ij} = \mathbf{d}^*_{ij}$ as in (13) for all $j \in \overline{\mathcal{N}}^i$ and for all $i \in \mathcal{V}$, is strongly positively correlated with $u(.)$. •

Although Lemma 5.3 allows us to directly apply most of the results in Section III, the description of NBRD through the optimization problem (11) poses a challenge for showing that the optimizer of $\{\mathbf{P}^i_2\}_{i \in \mathcal{V}}$, $\{\mathbf{d}^*_{ij}\}_{(i,j) \in \mathcal{A}}$, is a locally Lipschitz function of $\mathbf{x}$ as we do not have a closed-form expression for the optimizer. Hence, we resort to an in indirect approach. In the following lemma, we use Lemma 5.1 to show that the optimizer as a function of $\mathbf{x}$ is one of finitely many functions, which we show subsequently are all locally Lipschitz. This characterization allows us to apply Lemma 3.1 and establish existence and uniqueness of solutions for NBRD. The proof of the following lemma appears in Appendix B.

Lemma 5.4: (Computation of optimizers of $\boldsymbol{P}^i_2$ under a given flow graph). Let $\{\mathbf{d}^*_{ij}\}_{j \in \overline{\mathcal{N}}^i}$ be the optimizer of $\mathbf{P}^i_2$ for a given $\mathbf{x}$ and let $\mathcal{M}^i$ be the support of the optimizer. Consider the function $g_{\mathcal{M}^i}(.) : \mathbb{R} \to \mathbb{R}$ defined as

$$g_{\mathcal{M}^i}(y) := \sum_{j \in \mathcal{M}^i} u_j^{-1}(y). \tag{14}$$

Then η_i of Lemma 5.1 is given by

$$\eta_i = g^{-1}_{\mathcal{M}^i}\left(\sum_{k \in \mathcal{M}^i \cup \{i\}} \mathbf{x}_k\right). \tag{15}$$

Moreover, $\forall j \in \mathcal{M}^i \setminus \{i\}$

$$\mathbf{d}^*_{ij} = u_j^{-1}\left(g^{-1}_{\mathcal{M}^i}\left(\sum_{k \in \mathcal{M}^i \cup \{i\}} \mathbf{x}_k\right)\right) - \mathbf{x}_j, \tag{16a}$$

and

$$\mathbf{d}^*_{ii} = u_i^{-1}\left(g^{-1}_{\mathcal{M}^i}\left(\sum_{k \in \mathcal{M}^i \cup \{i\}} \mathbf{x}_k\right)\right), \quad \text{if } i \in \mathcal{M}^i. \tag{16b}$$

•

From the KKT conditions (28), notice that the multipliers μ^*_{ij} encode the support $\mathcal{M}^i$ while $\eta_i = \lambda^*_i$. Thus, the characterization of $\{\mathbf{d}^*_{ij}\}$ in Lemma 5.4 is closely related to solving the dual of problem $\mathbf{P}^i_2$. This alternate characterization of $\{\mathbf{d}^*_{ij}\}$ through the dual helps us in applying Lemma 3.1 in order to guarantee existence and uniqueness of solutions for NBRD. We now present the main result of this section where we show asymptotic convergence of solutions of NBRD to the set $\mathcal{NE}$. We provide the proof the result in Appendix B.

Theorem 5.5: (Existence and uniqueness of solutions for NBRD and asymptotic convergence to the Nash equilibrium set). Consider NBRD, the dynamics in (4) with $\delta_{ij} = \mathbf{d}^*_{ij}$ $\forall (i,j) \in \mathcal{A}$, where $\{\mathbf{d}^*_{ij}\}_{(i,j) \in \mathcal{A}}$ is the set of optimizers of the problems $\{\mathbf{P}^i_2\}_{i \in \mathcal{V}}$. For each $\mathbf{x}(0) \in \mathcal{S}$, NBRD has a unique solution that exists for $\forall t \geq 0$. The set of equilibrium points of NBRD in $\mathcal{S}$ is $\mathcal{NE}$ defined in (6). Further, if $\mathbf{x}$ evolves according to NBRD, then as $t \to \infty$, $U(\mathbf{x}(t))$ converges to a constant and $\mathbf{x}(t)$ approaches $\mathcal{NE}$. •

VI. Network Restricted Payoff Maximization

In this section, we analyze the dynamics arising out of centralized network restricted gradient ascent of the social utility, $U(.)$ given in (1). The inter-agent communication requirements for this dynamics depends on the particular application. The communication may be through broadcasting of signals or

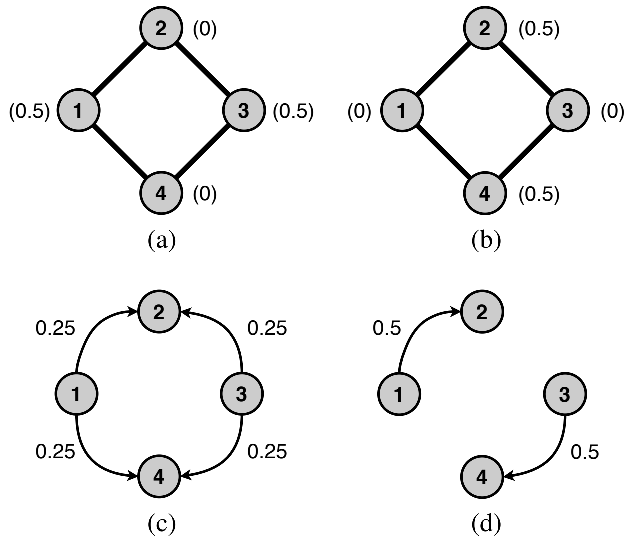


Fig. 1: Non-unique optimal outflow choice under NRPM. The number in (.) next to a node i represents the population fraction $\mathbf{x}_i$. The number near arc (i,j) represents $\mathbf{d}^*_{ij}$. (a) Initial population state (b) Optimal redistributed population state (c), (d) Different optimal outflow choices.

distributed gossip or there may be a centralized operator providing macro signals to the agents. The last mode of operation is relevant for swarm control and fleet redistribution by a taxi operator. We first present the underlying optimization problem and then formally define the dynamics. Specifically, the dynamics is (4) with the outflows $\delta_{ij} = \mathbf{d}_{ij}$, $\forall (i,j) \in \mathcal{A}$, which come from the optimizers of the problem $\mathbf{P}_3$, where

$$\mathbf{P}_3 :$$

$$\max_{\mathbf{z},\mathbf{d}} \sum_{i \in \mathcal{V}} [p_i(\mathbf{z}_i) - p_i(0)] \quad = \quad \max_{\mathbf{z},\mathbf{d}} U(\mathbf{z}) \tag{17a}$$

$$\text{s.t. } \mathbf{z}_i = \mathbf{x}_i + \sum_{j \in \overline{\mathcal{N}}^i} (\mathbf{d}_{ji} - \mathbf{d}_{ij}), \ \forall i \in \mathcal{V}, \tag{17b}$$

$$\sum_{j \in \overline{\mathcal{N}}^i} \mathbf{d}_{ij} = \mathbf{x}_i, \ \forall i \in \mathcal{V}, \tag{17c}$$

$$\mathbf{d}_{ij} \geq 0, \ \forall (i,j) \in \overline{\mathcal{A}} := \mathcal{A} \cup \{(i,i) \mid i \in \mathcal{V}\}. \tag{17d}$$

Note that although $U(\mathbf{z})$ is strictly concave in $\mathbf{z}$, the cost function in $\mathbf{P}_3$ is only concave in $(\mathbf{z}, \mathbf{d})$. Thus, in general, there may be more than one optimizer for $\mathbf{P}_3$. This is unlike the case for NBRD, where we have unique optimizers for $\mathbf{P}_2^i$, $\forall i \in \mathcal{V}$. We show an example in the following.

Example 6.1: (Non-unique optimal outflow choices under NRPM). Consider the graph with node set $\mathcal{V} = \{1,2,3,4\}$ and edge set $\mathcal{E} = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$. Let the cumulative payoff functions be $p_i(y) = -0.5\,y^2 - a_i\,y$. Thus, the payoff density functions are $u_i(y) = -y - a_i$. This is a uniform water-tank model as described in [24]. Let $a_1 = a_3 = 1$, $a_2 = a_4 = 0$ and $\mathbf{x} = [0.5, 0, 0.5, 0]^T$. A quick computation reveals that the optimal redistributed population state in this case is $\mathbf{z}^* = [0, 0.5, 0, 0.5]^T$. This can be done with multiple choices for $\mathbf{d}$, two of which are shown in Figure 1. •

However, for all the optimizers, the resultant node fraction $\mathbf{z}^*_i$, for each node $i \in \mathcal{V}$ is unique. This property is a direct application of the following lemma, whose proof is in Appendix D.

Lemma 6.2: (Uniqueness of a subset of optimizer variables in a class of convex optimization problems). Let $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{w} \in \mathbb{R}^m$. Consider the optimization problem

$$\max_{(\mathbf{v},\mathbf{w}) \in \Omega} f(\mathbf{v}).$$

Suppose that the set $\Omega \subset \mathbb{R}^{n+m}$ is non-empty and convex whereas the function $f(.)$ is a strictly concave function of $\mathbf{v}$. Then, every optimizer $(\mathbf{v}^i, \mathbf{w}^i)$ has the property that $\mathbf{v}^i = \mathbf{v}^*$, a unique constant. •

Direct application of Lemma 6.2 to problem $\mathbf{P}_3$ gives us the following result.

Lemma 6.3: (The resultant node fractions in any optimal solution of $\mathbf{P}_3$ are unique). Consider the problem $\mathbf{P}_3$ in (17) with $\mathbf{x} \in \mathcal{S}$. Let $\mathcal{OP}$ be the set of all optimizers of $\mathbf{P}_3$. Then $\forall\,(\mathbf{z}^*, \mathbf{d}^*) \in \mathcal{OP}$, $\mathbf{z}^*$ is unique. •

We make some more observations about the optimizers of $\mathbf{P}_3$ in the following lemma. Its proof is in Appendix C.

Lemma 6.4: (Properties of optimizers of $\mathbf{P}_3$). Suppose $(\mathbf{z}^*, \mathbf{d}^*)$ is an optimizer of $\mathbf{P}_3$. Then $\forall\,i \in \mathcal{V}$ there exists a constant $\overline{\eta}_i$ such that

$$u_j(\mathbf{z}^*_j) = \overline{\eta}_i, \quad \forall j \in \overline{\mathcal{N}}^i \text{ s.t. } \mathbf{d}^*_{ij} > 0\,. \tag{18}$$

Also,

$$u_j(\mathbf{z}^*_j) \geq u_i(\mathbf{z}^*_i), \ \forall j \in \mathcal{N}^i \text{ s.t. } \mathbf{d}^*_{ij} > 0\,. \tag{19}$$

•

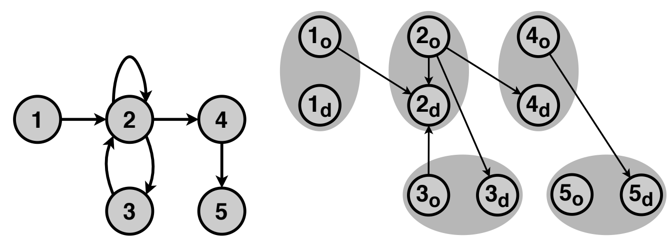
Remark 6.5: (Network restricted payoff maximization dynamics). Problem $\mathbf{P}_3$ gives the optimizer $\mathbf{z}^*$ of the social utility, $U(.)$, under the assumption that the agents may revise their choices to one of the nodes neighboring their current node, though with complete knowledge of the population state $\mathbf{x}$. Note that $\mathbf{P}_3$ is feasible for each $\mathbf{x} \in \mathcal{S}$ as $\mathbf{z}_i = \mathbf{d}_{ii} = \mathbf{x}_i$, $\forall i \in \mathcal{V}$ and $\mathbf{d}_{ij} = 0$, $\forall (i,j) \in \mathcal{A}$ is a feasible solution. Then, we let the evolution of $\mathbf{x}$ as a whole be (4) with $\delta_{ij} = \mathbf{d}^*_{ij}$ for all $(i,j) \in \mathcal{A}$, where $\{\mathbf{d}^*_{ij}\}_{(i,j)\in\mathcal{A}}$ comes from an arbitrary optimizer of $\mathbf{P}_3$. Note that, defined this way, even though we have non-uniqueness of δ_{ij}'s, the effect is unique as in Lemma 6.3, we have established that $\mathbf{z}^*(\mathbf{x}) = \mathbf{x} + \mathbf{J}\,\mathbf{d}^*(\mathbf{x})$ is uniquely defined for each $\mathbf{x} \in \mathcal{S}$. The resulting dynamics is

$$\dot{\mathbf{x}} = \mathbf{J}\,\Delta(\mathbf{x}) = \mathbf{z}^*(\mathbf{x}) - \mathbf{x}. \tag{20}$$

We refer to the dynamics (20) as *network restricted payoff maximization* (NRPM). •

Note that (19) is not the same as strong positive correlation of $\mathbf{J}\Delta(\mathbf{x})$ with $u(.)$ as in (19), the condition on $u(.)$ involves $\mathbf{z}^*$ instead of $\mathbf{x}$.

Example 6.6: (NRPM may not satisfy strong positive correlation with $u(.)$). Consider the graph with node set $\mathcal{V} = \{1,2,3\}$ and edge set $\mathcal{E} = \{\{1,2\},\{2,3\}\}$. Let the cumulative payoff functions be of the form $p_i(y) = -0.5\,y^2 - a_i\,y$ and hence the payoff density functions are of the form $u_i(y) = -y - a_i$. Let $a_1 = a_2 = 2$ and $a_3 = 0$ and let $\mathbf{x} = [0.2, 0.8, 0]^T$. For such a setup there is a unique optimizer



(a) (b)

Fig. 2: Construction of origin and destination sets. (a) The directed graph $(\mathcal{V}, \mathcal{M})$. (b) The constructed graph $(\hat{\mathcal{V}}, \hat{\mathcal{M}})$. The non-trivial weakly connected components are $\{1_o, 2_o, 3_o, 2_d, 3_d, 4_d\}$ and $\{4_o, 5_d\}$. Thus, the two sets of origin and destination nodes are $\mathcal{O}^1 = \{1, 2, 3\}$, $\mathcal{D}^1 = \{2, 3, 4\}$ and $\mathcal{O}^2 = \{4\}$, $\mathcal{D}^2 = \{5\}$.

$\mathbf{d}^*$ of $\mathbf{P}_3$. A quick computation will reveal that $\mathbf{d}^*_{12} = 0.2$ even though $u_1(0.2) > u_2(0.8)$. •

A. Existence and Uniqueness of Solutions for NRPM

Similar to NBRD, as $\mathbf{F}(.)$ for NRPM is obtained through an optimization problem, analyzing the dynamics directly from its definition can be cumbersome. In this subsection, we use Lemma 6.4 to analyze Lipschitzness of $\mathbf{z}^*(\mathbf{x})$. The idea is to appropriately partition the set of nodes into components on which $u_i(\mathbf{z}^*_i)$ is the same and then determining $\mathbf{z}^*$ using (18). This further leads to a covering of the simplex $\mathcal{S}$, where in each set of the covering, $\mathbf{z}^*(\mathbf{x})$ is locally Lipschitz so that we can again apply Lemma 3.1. Though this idea is similar to the one we used for NBRD, here the situation is complicated by the fact that there is coordination among the agents across the network. This leads to the construction being dependent on the network as a whole and not just the immediate neighborhood of each node, as in NBRD.

Construction of the origin-destination graph and the origin-destination sets: First consider the set $\overline{\mathcal{A}}$ and its power set $\mathcal{P}(\overline{\mathcal{A}}) := \{\mathcal{M} \mid \mathcal{M} \subseteq \overline{\mathcal{A}}\}$. We can compute candidate values of $\mathbf{z}^*$ assuming the support for the Lagrange multipliers (see (32)) ν^*_{ij}'s is $\mathcal{M}$, for each $\mathcal{M} \subseteq \overline{\mathcal{A}}$. Then, we can determine the set of $\mathbf{x}$ for which the support of ν^*_{ij} is indeed $\mathcal{M}$. For this purpose, we first consider the graph $(\mathcal{V}, \mathcal{M})$ and then construct a new graph $(\hat{\mathcal{V}}, \hat{\mathcal{M}})$ and pairs of subsets of nodes $(\mathcal{O}^p, \mathcal{D}^p)$ as follows.

- For each node $i \in \mathcal{V}$, we have two nodes in $\hat{\mathcal{V}}$, labeled i_o and i_d, standing for "origin" and "destination". For each arc $(i, j) \in \mathcal{M}$, we have an arc $(i_o, j_d) \in \hat{\mathcal{M}}$. If an arc $(i, i) \in \mathcal{M}$ then the corresponding arc in $\hat{\mathcal{M}}$ is (i_o, i_d). The graph $(\hat{\mathcal{V}}, \hat{\mathcal{M}})$ contains no other nodes or arcs.
- For each weakly connected component $(\hat{\mathcal{V}}_r, \hat{\mathcal{M}}_r)$ of the graph $(\hat{\mathcal{V}}, \hat{\mathcal{M}})$ that is non-trivial (having more than one node), we let origin and destination sets, $\mathcal{O}^r$ and $\mathcal{D}^r$, be

$$\mathcal{O}^r := \{i \in \mathcal{V} \mid i_o \in \hat{\mathcal{V}}_r\}, \quad \mathcal{D}^r := \{i \in \mathcal{V} \mid i_d \in \hat{\mathcal{V}}_r\}. \quad (21)$$

Figure 2 shows an example of this construction. This construction is useful in determining $\mathbf{z}^*$ if the support of $\{\mathbf{d}^*_{ij}\}_{(i,j)\in\overline{\mathcal{A}}}$ is $\mathcal{M}$. More precisely, we impose the conditions

$$\nu_{ij} = 0, \ \forall (i,j) \in \mathcal{M}, \quad \mathbf{d}_{ij} = 0, \ \forall (i,j) \notin \mathcal{M}. \quad (22)$$

Finally, given $\mathcal{M}$ and the resulting $\mathcal{O}^r$'s and $\mathcal{D}^r$'s, we define $F^{\mathcal{M}}(\mathbf{x}) : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ as

$$F^{\mathcal{M}}_i(\mathbf{x}) = \begin{cases} 0, & i \notin \bigcup_{r\in[1,n]_{\mathbb{Z}}} \mathcal{D}^r \\ u_i^{-1}\left(g^{-1}_{\mathcal{D}^r}\left(\sum_{i\in\mathcal{O}^r} \mathbf{x}_i\right)\right), & i \in \mathcal{D}^r, \end{cases} \quad (23)$$

with $g_{\mathcal{D}^r}(.)$ defined in (14). Here $n \le N$ is the number of non-trivial weakly connected components in $(\hat{\mathcal{V}}, \hat{\mathcal{M}})$. From the proof of Theorem 5.5, we know that $g^{-1}_{\mathcal{D}^r}(.)$ is continuously differentiable. Hence, we can say that $F^{\mathcal{M}}(\mathbf{x})$ is also a continuously differentiable function of $\mathbf{x}$. We are now ready to present the connection between this construction and the optimization problem $\mathbf{P}_3$. We provide its proof in Appendix C.

Lemma 6.7: (Optimizers of $\mathbf{P}_3$ with support pattern $\mathcal{M}$). Let $(\mathbf{z}^*, \mathbf{d}^*)$ be an optimizer of $\mathbf{P}_3$ with the support pattern (22) for some $\mathcal{M} \subseteq \overline{\mathcal{A}}$. Let $\{\mathcal{O}^r\}_{r\in[1,n]_{\mathbb{Z}}}$ and $\{\mathcal{D}^r\}_{r\in[1,n]_{\mathbb{Z}}}$ be defined as in (21) with n being the number of non-trivial weakly connected components in the graph $(\hat{\mathcal{V}}, \hat{\mathcal{M}})$. Then, the following are true.

(a) $\exists (i,j) \in \overline{\mathcal{A}} \cap \mathcal{M}$, for all $i \in \mathcal{V}$ such that $\mathbf{x}_i > 0$.
(b) Support of $\mathbf{x}$ is a subset of $\cup_{r\in[1,n]_{\mathbb{Z}}} \mathcal{O}^r$.
(c) $\mathbf{z}^*(\mathbf{x}) = F^{\mathcal{M}}(\mathbf{x})$, where $F^{\mathcal{M}}(.)$ is given in (23). •

One way to solve $\mathbf{P}_3$ would be to start off with a candidate $\mathcal{M} \subseteq \overline{\mathcal{A}}$, construct the $\mathcal{O}^r$'s and $\mathcal{D}^r$'s, compute $(\mathbf{z}, \mathbf{d})$ and among all such $(\mathbf{z}, \mathbf{d})$ that are feasible for $\mathbf{P}_3$, we can pick the ones that maximize the objective function of $\mathbf{P}_3$. While this procedure is long-winded and may not be useful if our only interest was to solve $\mathbf{P}_3$, it helps us in proving Lipschitzness of $\mathbf{z}^*(.)$ using Lemma 3.1 and as a consequence existence and uniqueness of solutions for NRPM. We summarize this in the following lemma, whose proof is in Appendix C.

Lemma 6.8: (Existence and uniqueness of solutions for NRPM). The state equation in (20) with an initial condition $\mathbf{x}(0) \in \mathcal{S}$ has a unique solution $\forall\, t \ge 0$. •

B. On the Convergence of NRPM

We now present the main result for NRPM (20) - that the set of equilibrium points of NRPM is $\mathcal{NE}$ and that all state trajectories starting in $\mathcal{S}$ converge to the equilibrium set asymptotically.

Theorem 6.9: (NRPM converges asymptotically to the Nash equilibrium set). For the dynamics in (20), $\mathcal{X}$ the set of equilibrium points in $\mathcal{S}$ is the set $\mathcal{NE}$ in (6). If $\mathbf{x}(0) \in \mathcal{S}$ then as $t \to \infty$, $U(\mathbf{x}(t))$ converges to a constant and $\mathbf{x}(t)$ approaches the set $\mathcal{NE}$ in (6).

Proof: Recall from Lemma 6.3 that for any optimizer $(\mathbf{z}^*(\mathbf{x}), \mathbf{d}^*(\mathbf{x}))$ of $\mathbf{P}_3$, $\mathbf{z}^*(\mathbf{x})$ is unique. Also, notice from (20) that if $\mathbf{x} \in \mathcal{X}$ then $\mathbf{z}^*(\mathbf{x}) = \mathbf{x}$. Thus, in order to study the equilibrium set, it suffices to determine the set of $\mathbf{x}$ for which $\mathbf{P}_3$ has the solution $\mathbf{z}^* = \mathbf{x}$ and $\mathbf{d}^*_{ij} = 0$, $\forall (i,j) \in \mathcal{A}$ and $\mathbf{d}^*_{ii} = \mathbf{x}_i$, $\forall i \in \mathcal{V}$. Now, consider an arbitrary but fixed $i \in \operatorname{supp}(\mathbf{x})$. Applying (32b)-(32c) for $(i,i) \in \overline{\mathcal{A}}$, we see that $\mu^*_i = \nu^*_{ii} = 0$. Now applying (32b)-(32c) for all $(i,j) \in \overline{\mathcal{A}}$, we see that $u_i(\mathbf{x}_i) = u_i(\mathbf{z}^*_i) \ge u_i(\mathbf{z}^*_j) = u_i(\mathbf{x}_j)$. This implies that $\mathbf{x} \in \mathcal{NE}$. Using similar arguments, we can also show that the converse is true, that is, if $\mathbf{x} \in \mathcal{NE}$ then $\mathbf{z}^*(\mathbf{x}) = \mathbf{x}$,

$\mathbf{d}^*_{ij} = 0$, $\forall (i,j) \in \mathcal{A}$ and $\mathbf{d}^*_{ii} = \mathbf{x}_i$, $\forall i \in \mathcal{V}$ is an optimal solution of $\mathbf{P}_3$. This proves the first part of the result.

Now, we prove convergence. First, notice that the simplex $\mathcal{S}$ is positively invariant under the dynamics (20). Note again that $\mathbf{P}_3$ is convex in $(\mathbf{z}, \mathbf{d})$ and the function $V(.) := -U(.)$ is strictly convex. Also, note that $\mathbf{z} = \mathbf{x}$, $\mathbf{d}_{ij} = 0$, $\forall (i,j) \in \mathcal{A}$, $\mathbf{d}_{ii} = \mathbf{x}_i$, $\forall i \in \mathcal{V}$ is a feasible solution for $\mathbf{P}_3$ for all $\mathbf{x} \in \mathcal{S}$. Further, as $\mathbf{z}^*(\mathbf{x})$ is unique by Lemma 6.3 for all the optimizers of $\mathbf{P}_3$, $U(\mathbf{z}^*(\mathbf{x})) \geq U(\mathbf{x})$ or equivalently,

$$V(\mathbf{z}^*(\mathbf{x})) \leq V(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{S}. \tag{24}$$

Now, as $V(.)$ is a twice differentiable strictly convex function,

$$\nabla V(\mathbf{x}).[\mathbf{z}^*(\mathbf{x}) - \mathbf{x}] + V(\mathbf{x}) \leq V(\mathbf{z}^*(\mathbf{x})),$$

which implies

$$\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}).\dot{\mathbf{x}} \leq V(\mathbf{z}^*(\mathbf{x})) - V(\mathbf{x}) \leq 0, \quad \forall \mathbf{x} \in \mathcal{S}.$$

Here, we use the fact that in (20), $\dot{\mathbf{x}} = \mathbf{z}^*(\mathbf{x}) - \mathbf{x}$ and (24). Thus $V(\mathbf{x}(t))$ and hence $U(\mathbf{x}(t))$ converges to a constant. Further, $\dot{V}(\mathbf{x}) = 0$ implies $V(\mathbf{x}) \leq V(\mathbf{z}^*(\mathbf{x}))$, which along with (24) means $V(\mathbf{z}^*(\mathbf{x})) = V(\mathbf{x})$. But recall that for each $\mathbf{x} \in \mathcal{S}$, $\mathbf{z}(\mathbf{x}) = \mathbf{x}$ is feasible for $\mathbf{P}_3$. Also, by Lemma 6.3 we know that for all optimizers $(\mathbf{z}^*(\mathbf{x}), \mathbf{d}^*(\mathbf{x}))$ of $\mathbf{P}_3$, $\mathbf{z}^*(\mathbf{x})$ is unique. Thus, it must be that $\mathbf{z}^*(\mathbf{x}) = \mathbf{x}$, that is $\mathbf{x} \in \mathcal{X} = \mathcal{NE}$. Then, by LaSalle's invariance principle [27], the dynamics in (20) converges to the set $\mathcal{NE}$. ∎

VII. Conclusion

We proposed three dynamics that model the evolution of a stratified population of optimum seeking agents under different levels of coordination. For the case with selfish agents, we generalized the standard Smith dynamics to our setting. For the case with nodal level coordination, we proposed the dynamics based on the best response of the fraction of population in a node. The third dynamics is a centralized dynamics with population level coordination achieved through network restricted gradient ascent of the social utility. For all three dynamics, we showed existence and uniqueness of solutions and also asymptotic convergence of the social utility to a constant and convergence of the population state to the set of Nash equilibria.

Future work includes extending this framework to incorporate nodal capacity constraints and establishing connections with opinion dynamics. We would also like to utilize the results to come up with efficient algorithms to compute the converging social utility. Further, we would like to allow for changes in the cumulative functions at rate much slower than that at which the social utility converges and explore control problems.

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APPENDIX

A. Proofs of Results on Strongly Positively Correlated Dynamics

1) Proof of Lemma 3.3: *(By contradiction)* Suppose $\exists\, n$ such that $\{1, \cdots, n\} \subseteq \mathcal{V}$ (possibly after relabeling nodes) and

- $\forall i \in [1, n-1]_{\mathbb{Z}}$, $i+1 \in \mathcal{N}^i$ and $1 \in \mathcal{N}^n$
- $\forall i \in [1, n-1]_{\mathbb{Z}}$, $\delta_{i(i+1)} > 0$ and $\delta_{n1} > 0$.

This describes a directed cycle in $\tilde{\mathcal{F}}$ with the nodes $(1, \cdots, n, 1)$. Then the assumption that $\mathbf{F}(.)$ is strongly positively correlated implies

$$u_1(\mathbf{x}_1) < \cdots < u_n(\mathbf{x}_n) < u_1(\mathbf{x}_1),$$

which is a contradiction. ■

2) Proof of Lemma 3.4: As $U(.)$ is a strictly concave function and $\mathcal{S}$ is a convex set, $\mathbf{P}_1$ has a unique optimizer $\mathbf{x}^* \in \mathcal{S}$. The Lagrangian for $\mathbf{P}_1$ is

$$L_1 = \sum_{i \in \mathcal{V}} p_i(\mathbf{x}_i) - \lambda \left(\sum_{i \in \mathcal{V}} \mathbf{x}_i - 1 \right) + \sum_{i \in \mathcal{V}} \mu_i \mathbf{x}_i,$$

where $\lambda \in \mathbb{R}$ and $\{\mu_i \geq 0\}_{i \in \mathcal{V}}$ are the Lagrange multipliers. From the KKT conditions for $\mathbf{P}_1$, we have

$$u_i(\mathbf{x}_i^*) - \lambda + \mu_i = 0, \ \forall i \in \mathcal{V}, \tag{25a}$$
$$\mu_i \mathbf{x}_i^* = 0, \ \forall i \in \mathcal{V}. \tag{25b}$$

Thus, we must have

$$\mu_i^* = 0, \quad u_i(\mathbf{x}_i^*) - \lambda^* = 0, \quad \forall i \in \operatorname{supp}(\mathbf{x}^*). \tag{26}$$

Now, consider an arbitrary but fixed $i \in \operatorname{supp}(\mathbf{x}^*)$. Combining (25a) and the fact that $\mu_j^* \geq 0$, we get

$$u_j(\mathbf{x}_j^*) - \lambda^* \leq 0, \quad \forall j \in \mathcal{N}^i. \tag{27}$$

Combining (26) and (27) we see that $\mathbf{x}^* \in \mathcal{NE}$ and as a result, $\mathcal{NE}$ is non-empty.

Finally, note that if $\mathbf{F}(.)$ is strongly positively correlated with $u(.)$ then the fact that $\mathcal{NE}$ is a subset of the equilibrium set $\mathcal{X}$ follows directly from Definition 3.2 and the assumption that $\delta_{ij} = 0 \ \forall j \in \mathcal{N}^i$, if $\mathbf{x}_i = 0$. This completes the proof. ■

B. Proofs of Results on NBRD

1) Proof of Lemma 5.1: The objective function in (11) is strictly concave in $\{\mathbf{d}_{ij}\}_{j \in \overline{\mathcal{N}}^i}$ and the problem (11) is a convex program. Moreover, as $\mathbf{x}_i \geq 0$, the problem $\mathbf{P}_2^i$ is always feasible since $\mathbf{d}_{ij} = 0$, $\forall j \in \mathcal{N}^i$ and $\mathbf{d}_{ii} = \mathbf{x}_i$, $\forall i \in \mathcal{V}$ is feasible. This justifies the uniqueness of optimizers [28]. The Lagrangian for $\mathbf{P}_2^i$ can be written as

$$L_2^i = \sum_{j \in \mathcal{N}^i} [p_j(\mathbf{x}_j + \mathbf{d}_{ij}) - p_j(\mathbf{x}_j)] + [p_i(\mathbf{d}_{ii}) - p_i(0)]$$
$$- \lambda_i \left(\sum_{j \in \overline{\mathcal{N}}^i} \mathbf{d}_{ij} - \mathbf{x}_i \right) + \sum_{j \in \overline{\mathcal{N}}^i} \mu_{ij}\, \mathbf{d}_{ij}\,,$$

where λ_i and $\{\mu_{ij} \geq 0\}_{j \in \overline{\mathcal{N}}^i}$ are the Lagrange multipliers. From the KKT conditions for $\mathbf{P}_2^i$, we have

$$u_i(\mathbf{d}_{ii}) - \lambda_i + \mu_{ii} = 0, \tag{28a}$$
$$u_j(\mathbf{x}_j + \mathbf{d}_{ij}) - \lambda_i + \mu_{ij} = 0, \ \forall j \in \mathcal{N}^i, \tag{28b}$$
$$\mu_{ij} \mathbf{d}_{ij} = 0, \ \forall j \in \overline{\mathcal{N}}^i. \tag{28c}$$

Now, $\forall j \in \overline{\mathcal{N}}^i$ such that $\mathbf{d}_{ij}^* > 0$, by (28c) we have $\mu_{ij}^* = 0$. Then we obtain (12a) and (12b) directly from (28a) and (28b), respectively by setting $\eta_i = \lambda_i^*$. Now $\forall j \in \overline{\mathcal{N}}^i$ such that $\mathbf{d}_{ij}^* = 0$, we have $\mu_{ij}^* \geq 0$. This along with (28a) and (28b) gives (12c) with $\eta_i = \lambda_i^*$. ■

2) Proof of Lemma 5.3: Consider an arbitrary but fixed $i \in \mathcal{V}$. Notice then from (12a) and (12c) that $u_i(\mathbf{d}_{ii}^*) \leq \eta_i$. Now, if $\mathbf{d}_{ij}^* > 0$ for some $j \in \mathcal{N}^i$ then $\mathbf{d}_{ii}^* < \mathbf{x}_i$ and as $u_j(.)$'s are strictly decreasing functions for all $j \in \mathcal{V}$, we have the following

$$u_j(\mathbf{x}_j) > u_j(\mathbf{x}_j + \mathbf{d}_{ij}^*) = \eta_i \geq u_i(\mathbf{d}_{ii}^*) > u_i(\mathbf{x}_i),$$

where the equality is due to (12b). Thus $\mathbf{d}_{ij}^* > 0$ implies $u_j(\mathbf{x}_j) > u_i(\mathbf{x}_i)$. Applying the contrapositive of this statement to all $(i, j) \in \mathcal{A}$ completes the proof. ■

3) Proof of Lemma 5.4: We can make the following observations from the meaning of support and Lemma 5.1.

- As $\mathcal{M}^i$ is the support of the optimizers of $\mathbf{P}_2^i$, $\mathbf{d}_{ik}^* = 0$, for all $k \notin \mathcal{M}^i$.
- If $i \in \mathcal{M}^i$ then $\mathbf{d}_{ii}^* = u_i^{-1}(\eta_i)$, otherwise $\mathbf{d}_{ii}^* = 0$.
- $\mathbf{d}_{ij}^* = u_j^{-1}(\eta_i) - \mathbf{x}_j$, $\forall j \in \mathcal{M}^i \setminus \{i\}$.

Then, by the feasibility conditions of $\mathbf{P}_2^i$ we have

$$\sum_{j \in \mathcal{M}^i} \mathbf{d}_{ij}^* = \mathbf{x}_i$$

which implies

$$\sum_{j \in \mathcal{M}^i} u_j^{-1}(\eta_i) = \mathbf{x}_i + \sum_{j \in \mathcal{M}^i \setminus \{i\}} \mathbf{x}_j\,. \tag{29}$$

Then (15) and (16) follow immediately from (29), the definition of the function $g_{\mathcal{M}^i}(.)$ in (14) and the observations about $\mathbf{d}_{ij}^*$ for $j \in \mathcal{M}^i$. Note that, as $u_i(.)$ is strictly decreasing and continuous for each i, so are $u_i^{-1}(.)$ and $g_{\mathcal{M}^i}(.)$. Thus the inverse function $g_{\mathcal{M}^i}^{-1}(.)$ exists. ■

4) Proof of Theorem 5.5: To show existence and uniqueness of solutions, we consider $\mathbb{R}_+^N$ to be the domain of $\Delta(\mathbf{x})$, instead of restricting $\mathbf{x}$ to be in $\mathcal{S}$. Consider the problem $\mathbf{P}_2^i$ for an arbitrary i. Suppose the support of $\{\mathbf{d}_{ij}^*\}_{j \in \overline{\mathcal{N}}^i}$ is known a priori. Then, the optimizer of $\mathbf{P}_2^i$ can be found easily using (16). Alternatively, we can solve the problem $\mathbf{P}_2^i$ by computing the candidate $\{\mathbf{d}_{ij}\}_{j \in \overline{\mathcal{N}}^i}$ using (16) for each candidate support $\mathcal{M}_s^i \subseteq \overline{\mathcal{N}}^i$ and picking the one that satisfies feasibility and maximizes the objective function of the problem $\mathbf{P}_2^i$.

In particular, given a candidate support $\mathcal{M}_s^i \subseteq \overline{\mathcal{N}}^i$, we let $\eta_i(\mathcal{M}_s^i)$ be as in (15) with $\mathcal{M}^i$ replaced by $\mathcal{M}_s^i$. Then, letting $w_j := u_j^{-1}(\eta_i(\mathcal{M}_s^i))$, the feasibility conditions of $\mathbf{P}_2^i$ can be re-written as

$$w_j - \mathbf{x}_j \geq 0, \quad \forall j \in \mathcal{M}_s^i \setminus \{i\} \tag{30a}$$
$$w_i \geq 0, \quad \text{if } i \in \mathcal{M}_s^i \tag{30b}$$
$$\sum_{j \in \mathcal{M}_s^i} w_j = \mathbf{x}_i + \sum_{j \in \mathcal{M}_s^i \setminus \{i\}} \mathbf{x}_j \tag{30c}$$
$$w_j \geq \mathbf{x}_j, \quad \forall j \in \overline{\mathcal{N}}^i \setminus \mathcal{M}_s^i. \tag{30d}$$

Note that the last condition is a re-expression of (12c). We can also write the objective function of $\mathbf{P}_2^i$ directly in terms of $\eta_i(\mathcal{M}_s^i)$ as

$$Q_{\mathcal{M}_s^i}(\mathbf{x}) := \sum_{j \in \mathcal{M}_s^i \setminus \{i\}} [p_j(w_j) - p_j(\mathbf{x}_j)] + l \left[p_i(w_i) - p_i(0) \right], \tag{31}$$

where

$$l = \begin{cases} 1, & \text{if } i \in \mathcal{M}_s^i \\ 0, & \text{otherwise}. \end{cases}$$

Thus, in the set

$$\mathcal{C}_{\mathcal{M}_r^i} := \left\{ \mathbf{x} \in \mathbb{R}_+^N \;\middle|\; Q_{\mathcal{M}_r^i}(\mathbf{x}) \geq Q_{\mathcal{M}_q^i}(\mathbf{x}), \ \forall \mathcal{M}_q^i \subseteq \overline{\mathcal{N}}^i \right.$$
$$\left. \text{s.t. (30) satisfied with } \mathcal{M}_s^i \text{ replaced by } \mathcal{M}_q^i \right\},$$

the choice of the support of the optimizer $\{\mathbf{d}_{ij}^*\}_{j \in \overline{\mathcal{N}}^i}$ is $\mathcal{M}^i = \mathcal{M}_r^i$ as the objective function of $\mathbf{P}_2^i$ is maximized and the feasibility constraints of $\mathbf{P}_2^i$ are satisfied. Now note that in this set $\mathcal{C}_{\mathcal{M}_r^i}$, $\{\mathbf{d}_{ij}^*\}_{j \in \overline{\mathcal{N}}^i}$ is given by (16a) with $\mathcal{M}^i = \mathcal{M}_r^i$. Since the payoff density functions $u_j(.)$'s are strictly decreasing and continuously differentiable, their derivatives are never zero. Hence by the inverse function theorem, their inverse functions are also continuously differentiable. This same reasoning can be done for the functions $g_{\mathcal{M}^i}(.)$ and $g_{\mathcal{M}^i}^{-1}(.)$. Thus δ_{ij} is continuously differentiable and hence locally Lipschitz in each $\mathcal{C}_{\mathcal{M}_r^i}$. Now as the set $\overline{\mathcal{N}}^i$ is finite, so is the set of all its subsets. Thus, the number of sets $\mathcal{C}_{\mathcal{M}_s^i}$ is also finite. Moreover, from their definition, these sets are also closed. Such coverings can be found $\forall i \in \mathcal{V}$. Thus, by Lemma 3.1, for each $\mathbf{x}(0) \in \mathcal{S}$, we have existence and uniqueness of solutions of NBRD $\forall t \geq 0$.

Now, by Lemma 5.3 we know that NBRD is strongly positively correlated with $u(.)$. Then, by Lemma 3.3 it is evident that the only $\Delta(\mathbf{x}) \in \ker(\mathbf{J})$ is $\Delta(\mathbf{x}) = \mathbf{0}$. Now if $\Delta(\mathbf{x}) = 0$ then $\forall i \in \operatorname{supp}(\mathbf{x})$, $\mathbf{d}_{ii}^* = \mathbf{x}_i > 0$. Then by combining (28a) with (28c) we get $u_i(\mathbf{x}_i) = \lambda_i^*$. Combining this with (28b) and the fact that $\mu_{ij}^* \geq 0$ we get

$$u_i(\mathbf{x}_i) \geq u_j(\mathbf{x}_j), \ \forall j \in \mathcal{N}^i, \ \forall i \in \operatorname{supp}(\mathbf{x}).$$

Thus, $\mathcal{NE} \supseteq \mathcal{X}$, the set of equilibrium points in $\mathcal{S}$. Further, from Lemma 3.4, we also know that $\mathcal{NE} \subseteq \mathcal{X}$ and that it is non-empty. Thus, for NBRD $\mathcal{X} = \mathcal{NE}$. Finally, application of Lemmas 5.3 and Theorem 3.7 completes the proof. ■

C. Proofs of Results on NRPM

1) Proof of Lemma 6.4: The lagrangian for $\mathbf{P}_3$ can be written as

$$L_3 = \sum_{i \in \mathcal{V}} p_i(\mathbf{z}_i) - \sum_{i \in \mathcal{V}} \lambda_i \left(\mathbf{z}_i - \mathbf{x}_i - \sum_{j \in \overline{\mathcal{N}}^i} (\mathbf{d}_{ji} - \mathbf{d}_{ij}) \right)$$
$$- \mu_i \left(\sum_{j \in \overline{\mathcal{N}}^i} \mathbf{d}_{ij} - \mathbf{x}_i \right) + \sum_{(i,j) \in \overline{\mathcal{A}}} \nu_{ij} \, \mathbf{d}_{ij},$$

where $\{\lambda_i\}_{i \in \mathcal{V}}$, $\{\mu_i\}_{i \in \mathcal{V}}$ and $\{\nu_{ij} \geq 0\}_{(i,j) \in \overline{\mathcal{A}}}$ are the Lagrange multipliers. Then, from the KKT conditions for $\mathbf{P}_3$, we have

$$u_i(\mathbf{z}_i) - \lambda_i = 0, \ \forall i \in \mathcal{V}, \tag{32a}$$
$$- \lambda_i + \lambda_j - \mu_i + \nu_{ij} = 0, \ \forall (i,j) \in \overline{\mathcal{A}}, \tag{32b}$$
$$\nu_{ij} \, \mathbf{d}_{ij} = 0, \ \forall (i,j) \in \overline{\mathcal{A}}. \tag{32c}$$

Consider an arbitrary but fixed $i \in \mathcal{V}$. If $\mathbf{d}_{ij}^* > 0$ for only one $j \in \overline{\mathcal{N}}^i$ then (18) is trivially true. So now, suppose that $\mathbf{d}_{ij}^* > 0$ and $\mathbf{d}_{ik}^* > 0$ for $j \neq k$ and $j, k \in \overline{\mathcal{N}}^i$. Then by (32c), $\nu_{ij}^* = \nu_{ik}^* = 0$ which, by (32a) and (32b), gives

$$u_j(\mathbf{z}_j^*) = u_k(\mathbf{z}_k^*) = \lambda_j = \lambda_k = \lambda_i + \mu_i. \tag{33}$$

This proves (18).

Now, if $\mathbf{d}_{ii}^* > 0$ and $\mathbf{d}_{ij}^* > 0$ for some $j \in \overline{\mathcal{N}}^i$ then by (33), we see that (19) is true. So, now suppose that $\mathbf{d}_{ii}^* = 0$ and $\mathbf{d}_{ij}^* > 0$ for some $j \in \overline{\mathcal{N}}^i$. Then, (32b) and (32c) imply that $\mu_i = \nu_{ii} \geq 0$. Now, using (32b) and (32c) for (i,j), we have $\lambda_j = \lambda_i + \mu_i \geq \lambda_i$. Thus, we now obtain (19) by using (32a) for i and j. ■

2) Proof of Lemma 6.7: **(a):** This claim follows immediately from the feasibility constraints (17c) and (17d) in $\mathbf{P}_3$.

(b): This claim now follows from **(a)** and the construction of the $\mathcal{O}^r$ sets.

(c): Notice from (21) that if $i \notin \cup_{r \in [1,n]_{\mathbb{Z}}} \mathcal{D}^r$ then $\nexists (k,i) \in \mathcal{M}$. Then, from (22), we see that $\mathbf{d}_{ki}^* = 0$ for all $(k,i) \in \overline{\mathcal{A}}$, which along with (17b)- (17d), implies that $\mathbf{z}_i^* = 0$. Thus, for all $i \notin \cup_{r \in [1,n]_{\mathbb{Z}}} \mathcal{D}^r$, we have $\mathbf{z}_i^* = F_i^{\mathcal{M}}(\mathbf{x})$.

Now, from the construction of $\mathcal{O}^r$ and $\mathcal{D}^r$ and the support pattern (22), it is evident that

$$\sum_{i \in \mathcal{D}^r} \mathbf{z}_i^* = \sum_{i \in \mathcal{O}^r} \mathbf{x}_i, \ \forall r \in [1,n]_{\mathbb{Z}}. \tag{34}$$

Further, applying Lemma 6.4 with the support pattern (22), and the definition of $\mathcal{D}^r$, we see that

$$u_i(\mathbf{z}_i^*) = \overline{\eta}_{\mathcal{D}^r}, \ \forall i \in \mathcal{D}^r, \ \forall r \in [1,n]_{\mathbb{Z}}, \tag{35}$$

for some $\overline{\eta}_{\mathcal{D}^r}$. Now, considering (34)-(35) together gives us

$$\overline{\eta}_{\mathcal{D}^r} = g_{\mathcal{D}^r}^{-1} \left(\sum_{i \in \mathcal{O}^r} \mathbf{x}_i \right), \forall r \in [1,n]_{\mathbb{Z}},$$

from which claim (c) now follows. ■

3) Proof of Lemma 6.8: Here, as in the proof of Theorem 5.5, instead of restricting $\mathbf{x}$ to $\mathcal{S}$, we consider $\mathbb{R}_+^N$ to be the domain of $\mathbf{z}^*(\mathbf{x})$. For each $\mathcal{M} \subseteq \overline{\mathcal{A}}$, $F^{\mathcal{M}}(\mathbf{x})$ is a candidate for $\mathbf{z}^*(\mathbf{x})$. Of course, $F^{\mathcal{M}}(\mathbf{x})$ may not even be a feasible value of $\mathbf{z}$ in $\mathbf{P}_3$. Thus, as a function of $\mathbf{x}$, we let the *set of feasible support patterns* be

$$\mathcal{F}(\mathbf{x}) := \{\mathcal{M} \subseteq \overline{\mathcal{A}} \mid \exists\, \mathbf{d} \text{ s.t. } \mathbf{z} = F^{\mathcal{M}}(\mathbf{x}),\ (17b) - (17d)\}.$$

For each $\mathbf{x} \in \mathbb{R}_+^N$, each optimizer $(\mathbf{z}^*, \mathbf{d}^*)$ of $\mathbf{P}_3$, must satisfy the support pattern (22) for some $\mathcal{M} \subseteq \overline{\mathcal{A}}$. Thus, for each $\mathbf{x} \in \mathbb{R}_+^N$, $\mathbf{z}^*(\mathbf{x}) = F^{\mathcal{M}}(\mathbf{x})$ for some $\mathcal{M} \subseteq \overline{\mathcal{A}}$. Further, for a given $\mathcal{M} \subseteq \overline{\mathcal{A}}$, $\mathbf{z}^*(\mathbf{x}) = F^{\mathcal{M}}(\mathbf{x})$ if $\mathbf{x} \in \mathcal{C}_{\mathcal{M}}$, where

$$\begin{aligned}\mathcal{C}_{\mathcal{M}} := \{\mathbf{x} \in \mathbb{R}_+^N \mid\ & \mathcal{M} \in \mathcal{F}(\mathbf{x}),\\ & U(F^{\mathcal{M}}(\mathbf{x})) \geq U(F^{\mathcal{M}'}(\mathbf{x})),\ \forall \mathcal{M}' \in \mathcal{F}(\mathbf{x})\}.\end{aligned}$$

Now, we seek to establish that if $\mathcal{C}_{\mathcal{M}}$ is non-empty then it is a closed subset of $\mathbb{R}_+^N$. Since $F^{\mathcal{M}}(.)$ is a continuously differentiable function for each $\mathcal{M} \subseteq \overline{\mathcal{A}}$ and since $U(.)$ is a strictly concave twice-continuously differentiable function, it suffices to show that

$$\begin{aligned}\mathcal{R}_{\mathcal{M}} &:= \{\mathbf{x} \in \mathbb{R}_+^N \mid \mathcal{M} \in \mathcal{F}(\mathbf{x})\}\\ &= \{\mathbf{x} \in \mathbb{R}_+^N \mid \exists\, \mathbf{d} \text{ s.t. } \mathbf{z} = F^{\mathcal{M}}(\mathbf{x}),\ (17b) - (17d)\}\end{aligned}$$

is closed. Notice that the constraints (17b)-(17d) with $\mathbf{z} = F^{\mathcal{M}}(\mathbf{x})$ are of the form $\mathbf{M}\mathbf{d} = f(\mathbf{x})$ and $\mathbf{d} \geq \mathbf{0}$ for some matrix $\mathbf{M}$ and $f(.)$, a continuously differentiable function of $\mathbf{x}$. From this, we can reason that every boundary point of $\mathcal{R}_{\mathcal{M}}$ must belong to $\mathcal{R}_{\mathcal{M}}$. Thus, $\mathcal{R}_{\mathcal{M}}$ and $\mathcal{C}_{\mathcal{M}}$ are both closed.

As there are finitely many subsets $\mathcal{M}$ of $\overline{\mathcal{A}}$, we have finitely many $\mathcal{C}_{\mathcal{M}}$'s which cover $\mathbb{R}_+^N$ and in each $\mathcal{C}_{\mathcal{M}}$, $F^{\mathcal{M}}(.)$ is a continuously differentiable function of $\mathbf{x}$. Thus, we can apply Lemma 3.1, to conclude that for each $\mathbf{x}(0) \in \mathcal{S}$, we have existence and uniqueness of solutions for NRPM $\forall t \geq 0$. ∎

D. Proofs of Auxiliary Results

1) Proof of Lemma 3.1: Consider an $\mathbf{x} \in \mathbb{R}_+^N$ and a ball $\mathcal{B}_2(\mathbf{x}, r)$ around $\mathbf{x}$ of some radius $r > 0$. Let L be a Lipschitz constant for $\mathbf{F}^i$ on the compact set $\mathcal{B}_2(\mathbf{x}, r) \cap \mathcal{C}_i$, for all $i \in \{1, \ldots, n\}$. Now, given $\mathbf{y}^1, \mathbf{y}^2 \in \mathcal{B}_2(\mathbf{x}, r)$, we construct a sequence of points $\{\mathbf{x}^j\}_{j \in \{0,1,\ldots,R\}}$ with increasing distance from $\mathbf{y}^1$, on the line joining $\mathbf{y}^1$ and $\mathbf{y}^2$ with $\mathbf{x}^0 = \mathbf{y}^1$ and $\mathbf{x}^R = \mathbf{y}^2$. We choose these points such that $\mathbf{x}^j, \mathbf{x}^{j+1} \in \mathcal{B}_2(\mathbf{x}, r) \cap \mathcal{C}_{r_j}$ for each j. Such a construction is possible since $\mathcal{C}_i$'s are all closed and $\bigcup_{i \in [1,n]_{\mathbb{Z}}} \mathcal{C}_i = \mathbb{R}_+^N$. Now, notice that

$$\begin{aligned}\|\mathbf{F}(\mathbf{y}^1) - \mathbf{F}(\mathbf{y}^2)\| &= \left\| \sum_{j=1}^{R} \mathbf{F}^{r_j}(\mathbf{x}^{j+1}) - \mathbf{F}^{r_j}(\mathbf{x}^j) \right\|\\ &\leq \sum_{j=1}^{R} \|\mathbf{F}^{r_j}(\mathbf{x}^{j+1}) - \mathbf{F}^{r_j}(\mathbf{x}^j)\|\\ &\leq \sum_{j=1}^{R} L\|\mathbf{x}^{j+1} - \mathbf{x}^j\| = L\|\mathbf{y}^2 - \mathbf{y}^1\|,\end{aligned}$$

where the first and the last equalities follows from the construction of $\{\mathbf{x}^j\}$ sequence. Since this holds for an arbitrary $\mathbf{x} \in \mathbb{R}_+^N$, $\mathbf{F}$ is locally Lipschitz on $\mathbb{R}_+^N$.

Now, note that $\mathcal{S}$ is a compact subset of $\mathbb{R}_+^N$ and $\mathbf{1}^T\dot{\mathbf{x}} = \mathbf{1}^T \mathbf{J}\,\Delta = \mathbf{0}$ as the rows of $\mathbf{J}$ sum to $\mathbf{0} \in \mathbb{R}^{2M}$. Thus, $\mathbf{F}$ is Lipschitz in $\mathcal{S}$ and $\mathcal{S}$ is positively invariant under (4). This guarantees existence and uniqueness of solutions $\forall t \geq 0$. ∎

2) Proof of Lemma 6.2: Let $\mathbf{q}^1 = (\mathbf{v}^1, \mathbf{w}^1)$ and $\mathbf{q}^2 = (\mathbf{v}^2, \mathbf{w}^2)$ be two optimizers with $\mathbf{v}^1 \neq \mathbf{v}^2$. Since $\mathbf{q}^1$ and $\mathbf{q}^2$ are optimizers, it must be that

$$f(\mathbf{v}^1) = f(\mathbf{v}^2).$$

Now, consider a convex combination $(\mathbf{v}, \mathbf{w}) =: \mathbf{q} = \sigma\mathbf{q}^1 + (1-\sigma)\mathbf{q}^2$ with $\sigma \in (0, 1)$. Since Ω is convex, $\mathbf{q} \in \Omega$ and

$$\begin{aligned}f(\mathbf{v}) &= f(\sigma\mathbf{v}^1 + (1-\sigma)\mathbf{v}^2)\\ &> \sigma f(\mathbf{v}^1) + (1-\sigma) f(\mathbf{v}^2) = f(\mathbf{v}^1) = f(\mathbf{v}^2),\end{aligned}$$

where the inequality follows from the strict concavity of $f(.)$. However, this means that $f(\mathbf{v}^1) = f(\mathbf{v}^2)$ is not the optimum value and hence $\mathbf{q}^1$ and $\mathbf{q}^2$ are not optimizers. This is a contradiction and hence the result must be true. ∎

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