A Control Lyapunov Function Approach to Event-Triggered Parameterized Control for Discrete-Time Linear Systems

Anusree Rajan, Kushagra Parmeshwar, and Pavankumar Tallapragada

Abstract— **This paper proposes an event-triggered parameterized control method using a control Lyapunov function approach for discrete time linear systems with external disturbances. In this control method, each control input to the plant is a linear combination of a fixed set of linearly independent scalar functions. The controller updates the coefficients of the parameterized control input in an event-triggered manner so as to minimize a quadratic cost function subject to quadratic constraints and communicates the same to the actuator. We design an event-triggering rule that guarantees global uniform ultimate boundedness of trajectories of the closed loop system and non-trivial inter-event times. We illustrate our results through numerical examples and we also compare the performance of the proposed control method with other existing control methods in the literature.**

I. INTRODUCTION

Event-triggered control (ETC) is a promising control method, especially in networked control systems, due to its efficient utilization of resources compared to the classical time-triggered control method. Recent studies in the ETC literature try to explore the possibility of further improving the efficiency of resource utilization by designing control laws based on non-zero order hold (non-ZOH) techniques instead of the popular ZOH technique, in which the control input to the plant is held constant between two successive communication times. However, most of the existing ETC methods based on non-ZOH control either require more computational capacity at the actuator or require transmitting a larger amount of information over the communication network at each communication time instant. An exception to this is the event-triggered parameterized control (ETPC) method proposed in [1]. In this paper, we extend this idea using a control Lyapunov function (CLF) method for discrete-time linear systems with external disturbances. This is in contrast to the emulation based approach, which is far more common in event-triggered control literature.

A. Literature Review

A fundamental overview of the ETC method, along with relevant literature, is discussed in [2]–[5]. Generally, in ETC and in other closely related approaches, such as self-triggered control [6] and periodic event-triggered control [7], the control input to the plant is held constant between any two consecutive triggering instants. However, there are some exceptions to this basic approach. For example, in model-based ETC [8]–[12], a time-varying control input is applied to the plant even between two successive events by using a model of the plant at the actuator. In event/self-triggered model predictive control (MPC) [13]–[15], at each triggering instant, the controller generates a control trajectory by solving a finite horizon optimization problem and then transmits it to the actuator, and the actuator applies the same to the plant until the next event. As discussed in [16], [17], the efficiency of communication resource utilization in MPC can be improved by transmitting only some of the samples of the generated control trajectory to the actuator, based on which a sampled data first-order-hold (FOH) control input is applied to the plant. Similar to the event-triggered MPC method, in event-triggered dead-beat control [18], a sequence of control inputs is transmitted to the actuator in an event-triggered manner and the same is applied to the plant till the next packet is received.

Our recent work [1] proposes a novel non-ZOH based ETC method, called as event-triggered parameterized control (ETPC) method, for stabilization of linear systems. In [19], we extend this control method to nonlinear control settings with external disturbances. In [20], we use a similar idea to design an event-triggered polynomial controller for trajectory tracking by unicycle robots. In all these works, we use an emulation based approach for determining the parameters at each event-triggering instant. There are also a few papers that use a parameterized control law in MPC like problems but not with even-triggering. For example, in our recent work [21], we co-design a polynomial control law and a communication scheduling strategy for multi-loop networked control systems. Another example is [22] which introduces a numerical algorithm that serves as a preliminary step toward solving continuous-time MPC problems directly without explicit time-discretization.

B. Contributions

The contributions of this paper are given below:

- We design an event-triggered parameterized control law for discrete-time linear systems with external disturbances, using a control Lyapunov function approach. For our proposed method, we guarantee global uniform ultimate boundedness of trajectories of the closed loop system and non-trivial inter-event times.
- In this paper, we extend the control method proposed in our previous works [1], [19], [20] to design an optimal control law for discrete-time linear systems with external disturbances. References [1], [19], [20] use an emulation based approach for designing the parameterized control law, where first a system model is simulated for some time duration in the future and

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then an ideal feedback control signal is approximated optimally using a parameterized control. Whereas, this paper uses a control Lyapunov function approach for designing the parameterized control law, where the control trajectory is directly optimized in the space of the parameterized functions.

- Compared to the model-based control method, the proposed parameterized control method requires less computational resources at the actuator and also provides greater privacy and security.
- Compared to the MPC-based control method, at each event, our proposed method requires only a limited number of parameters to be sent irrespective of the time duration of the signal.

C. Notation

Let $\mathbb{R}$ denote the set of all real numbers. Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all integers, positive and non-negative integers, respectively. For $a,b \in \mathbb{R}$, let $[a,b]_{\mathbb{Z}} := [a,b] \cap \mathbb{Z}$ and $[a,b)_{\mathbb{Z}} := [a,b) \cap \mathbb{Z}$. For any $x \in \mathbb{R}^n$, $\|x\|$ denotes the euclidean norm. For a square matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues, let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and the largest eigenvalues of A, respectively. Further, for a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $A \succ 0$, $A \succeq 0$ and $A \prec 0$ mean that A is positive definite, positive semi-definite and negative definite, respectively.

II. PROBLEM SETUP

This section describes the system dynamics, the parameterized control law and the objective of this paper.

System Dynamics and Control Law

Consider a discrete-time linear time-invariant system with external disturbance,

$$x(t+1) = Ax(t) + Bu(t) + d(t), \quad \forall t \in \mathbb{N}_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $d \in \mathbb{R}^n$, respectively, denote the system state, the control input, and the external disturbance.

(A1) Assume that there exists $D \geq 0$ such that $\|d(t)\| \leq D$, $\forall t \in \mathbb{N}_0$.

Consider a parameterized controller, where each control input to the plant is a linear combination of a set of linearly independent scalar functions. The linear combination is updated in an event-triggered manner. Specifically, we consider a set of functions

$$\Phi := \left\{\phi_j : \mathbb{N}_0 \to \mathbb{R}\right\}_{j=0}^{p},$$

which satisfies the following standing assumption.

(A2) Φ is a set of linearly independent functions when restricted to $[0,N]_{\mathbb{Z}}$ where $N \in \mathbb{N}$ is a fixed parameter, i.e., $\sum_{j=0}^{p} c_j \phi_j(t) = 0$, $\forall t \in [0,N]_{\mathbb{Z}}$ iff $c_j = 0$, $\forall j \in \{0,1,\ldots,p\}$.

Then, we consider the following control law,

$$u(t_k+\tau) = \mathbb{P}(\tau)a(k), \ \forall \tau \in [0, t_{k+1}-t_k)_{\mathbb{Z}}, \tag{2}$$

where

$$\mathbb{P}(\tau) := \begin{bmatrix} \phi^\top(\tau) & 0 & \ldots & 0 \\ 0 & \phi^\top(\tau) & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \phi^\top(\tau) \end{bmatrix} \in \mathbb{R}^{m\times m(p+1)},$$

and $\phi^\top(\tau) := \begin{bmatrix}\phi_0(\tau) & \phi_1(\tau) & \ldots & \phi_p(\tau)\end{bmatrix}$. Here, $a(k) \in \mathbb{R}^{m(p+1)}$ is a column vector which contains the coefficients of the parameterized control law. $(t_k)_{k\in\mathbb{N}_0}$ denotes the sequence of time instants at which the controller computes the coefficients of the parameterized control law and communicates them to the actuator. Note that, Assumption **(A2)** ensures that there exists a unique choice of coefficients for a desired control signal in the span of Φ.

The general configuration of the event-triggered parameterized control system considered in this paper is depicted in Figure 1. Here, the system state is continuously available

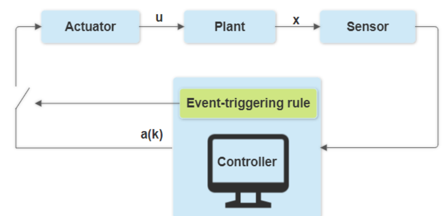


Fig. 1: Event-triggered parameterized control configuration

to the controller which has enough computational resources to evaluate the event-triggering condition and to update the coefficients of the control input at an event-triggering instant.

Objective

The main objective of this paper is to design a parameterized control law (2) using a control Lyapunov function approach, instead of the much more commonly used emulation based approach, and an event-triggering rule for implicitly determining the communication instants $(t_k)_{k\in\mathbb{N}_0}$ so that the trajectories of the closed loop system are globally uniformly ultimately bounded.

III. DESIGN OF EVENT-TRIGGERED CONTROLLER

This section discusses the design of a parameterized control law and an event-triggering rule to achieve the objective.

A. Design of Parameterized Control Law

At each triggering instant t_k, the controller determines the new coefficients $a(k)$ by solving the following finite horizon optimization problem,

$$\begin{aligned} a(k) \in \underset{a\in\mathbb{R}^{m(p+1)}}{\arg\min} \ & \sum_{t=t_k}^{t_k+N} \left[V(\hat{x}(t)) + u(t)^T R u(t)\right], \\ \text{s.t. } \ & \hat{x}(t+1) = A\hat{x}(t) + Bu(t), \ \hat{x}(t_k) = x(t_k), \\ & u(t) = \mathbb{P}(t-t_k)a, \ \forall t \in [t_k, t_k+N]_{\mathbb{Z}}, \\ & V(\hat{x}(t)) \leq \alpha^{t-t_k} V(\hat{x}(t_k)), \ \forall t \in [t_k, t_k+M]_{\mathbb{Z}}. \end{aligned} \tag{3}$$

Here, $M \leq N \in \mathbb{N}$ and $\alpha \in (0,1)$ are design parameters. $R \succeq 0$ and $V(x) := x^\top P x$ is a Lyapunov-like function with $P \succ 0$. Note that, the optimization problem (3) may have more than one optimizer and one among them is chosen as $a(k)$.

Note also that, given the dynamics, the closed form expression for $\hat{x}(t)$ can be written as follows,

$$\hat{x}(t) = F(t-t_k)x(t_k) + G(t-t_k)a, \ \forall t \in [t_k, t_k+N]_{\mathbb{Z}},$$

where $F(\tau) := A^\tau$ and $G(\tau) := \sum_{j=0}^{\tau-1} A^{\tau-1-j} B\mathbb{P}(j)$. By using the above expression, the optimization problem (3) can be rewritten as the following quadratically constrained quadratic optimization problem,

$$\begin{aligned} a(k) \in \underset{a\in\mathbb{R}^{m(p+1)}}{\arg\min} \quad & J(a), \\ \text{s.t.} \quad & H_\tau(a) < 0,\ \forall \tau \in [0,M]_{\mathbb{Z}}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} J(a) = a^\top & \left[\sum_{\tau=0}^{N} \left(G^\top(\tau)PG(\tau) + \mathbb{P}^\top(\tau)R\mathbb{P}(\tau)\right)\right] a \\ & + 2x^\top(t_k)\left[\sum_{\tau=0}^{N} F^\top(\tau)PG(\tau)\right] a \\ & + x^\top(t_k)\left[\sum_{\tau=0}^{N} F^\top(\tau)PF(\tau)\right] x(t_k), \end{aligned}$$

$$\begin{aligned} H_\tau(a) = a^\top G^\top(\tau)PG(\tau)a + 2x^\top(t_k)F^\top(\tau)PG(\tau)a \\ + x^\top(t_k)\left[F^\top(\tau)PF(\tau) - \alpha^\tau P\right] x(t_k). \end{aligned}$$

Remark 1. *The quadratically constrained quadratic optimization problem (4) is a convex optimization problem as $G^\top(\tau)PG(\tau) \succeq 0$ and $\mathbb{P}^\top(\tau)R\mathbb{P}(\tau) \succeq 0$ for all $\tau \in [0,N]_{\mathbb{Z}}$.* •

Remark 2. *In the emulation based approach used in [1], [19], [20], we first simulate the system for some time duration in the future and then optimally approximate the ideal feedback control signal using a parameterized control. In contrast, in this paper, we use a control Lyapunov function approach for designing the parameterized control law. The major advantages of this approach are that it directly optimizes the control trajectory in the space of the parameterized functions and we can also incorporate a greater variety of cost functions and constraints in the problem.* •

Remark 3. *Compared to the event-triggered model predictive control and dead-beat control, the ETPC method allows for having inter-event times strictly greater than the prediction horizon length N. Once the coefficients $a(k)$ are determined at t_k, the control law $u(t)$ is well defined for the interval $[t_k, t_{k+1})_{\mathbb{Z}}$ even if $t_{k+1} - t_k > N$. In addition, in ETPC, since only the parameters of the control signal need to be communicated, the communication load is significantly decreased even for long horizons N.* •

Remark 4. *Compared to the model-based control methods, the proposed method provides greater privacy and security as it requires a model of the system only at the controller and not at the actuator.* •

Next we provide a sufficient condition to ensure the feasibility of (4).

Proposition 5. *(Sufficient condition to ensure the feasibility of (4)). The optimization problem (4) is feasible if there exists a solution $C \in \mathbb{R}^{m(p+1)\times n}$ for the following linear matrix inequality (LMI), $\forall \tau \in [0,M]_{\mathbb{Z}}$,*

$$L_0(\tau) + \sum_{i,j} c_{ij} L_{ij}(\tau) \succ 0, \tag{5}$$

where c_{ij} denotes the $\{i,j\}^{th}$ element of C,

$$L_0(\tau) = \begin{bmatrix} \alpha^\tau P^{-1} & (F(\tau)P^{-1})^\top \\ F(\tau)P^{-1} & P^{-1} \end{bmatrix},$$

and

$$L_{ij} = \begin{bmatrix} 0 & (Q_{ij}(\tau))^\top \\ Q_{ij}(\tau) & 0 \end{bmatrix}.$$

Here, $Q_{ij}(\tau)$ is the matrix formed by multiplying the i^{th} column of $G(\tau)$ with the j^{th} row of P^{-1}.

Proof. First, note that, the linear matrix inequality (5) can be rewritten in the following matrix form,

$$\begin{bmatrix} \alpha^\tau P^{-1} & \left[\left(F(\tau) + G(\tau)C\right)P^{-1}\right]^\top \\ \left(F(\tau) + G(\tau)C\right)P^{-1} & P^{-1} \end{bmatrix} \succ 0.$$

Now, by using Schur complement lemma, we can say that the above inequality is true if and only if,

$$\alpha^\tau P^{-1} - \left[\left(F(\tau) + G(\tau)C\right)P^{-1}\right]^\top P \left[\left(F(\tau) + G(\tau)C\right)P^{-1}\right] \succ 0.$$

This implies that,

$$[F(\tau) + G(\tau)C]^\top P [F(\tau) + G(\tau)C] - \alpha^\tau P \prec 0.$$

If there exists a $C \in \mathbb{R}^{m(p+1)\times n}$ which satisfies the above inequality for all $\tau \in [0,M]_{\mathbb{Z}}$, then we can say that $a = Cx(t_k)$ is a feasible solution of the optimization problem (4). ☐

Remark 6. *Assume that ϕ_0 is a non-zero constant function and $\phi_j(0) = 0$, $\forall j \in \{1,2,\ldots,p\}$. If the pair (A,B) is controllable, then there always exists an $M \in \mathbb{N}$ and $C \in \mathbb{R}^{m(p+1)\times n}$ that satisfy the LMI (5) for the choice of $P \succ 0$ which is a solution of the Lyapunov equation $(A+BK)^\top P(A+BK) - P = -Q$ and $\alpha \in [1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}, 1)$, for some $Q \succ 0$ and $K \in \mathbb{R}^{m\times n}$ such that $A+BK$ is Schur stable and which satisfies the desired convergence rate constraint. Specifically $C \in \mathbb{R}^{m(p+1)\times n}$ such that $\mathbb{P}(0)C = K$ is guaranteed to satisfy the LMI (5) for $M = 1$.* •

B. Design of Event-Triggering Rule

Next, let us design an event-triggering rule that implicitly determines the time instants at which the controller updates the coefficients of the parameterized control input and communicates the same to the actuator. But first, we define the following *predictor* function

$$\begin{aligned} \bar{V}(t+1|t_k) :=& [Ax(t) + Bu(t|t_k)]^\top P [Ax(t) + Bu(t|t_k)] \\ & + \lambda_{\max}(P)\left(D^2 + 2D\left\|Ax(t) + Bu(t|t_k)\right\|\right), \end{aligned}$$

where $u(t|t_k) := \mathbb{P}(t - t_k)a(k)$ is the control trajectory computed at t_k. As we will see in the next result, the predictor function provides an upper bound on $V(x(t+1))$, over all possible disturbances, if at time t the control input $u(t) = u(t|t_k)$. Thus, the predictor function could help us evaluate the necessity of replanning and updating the control trajectory at each timestep.

Lemma 7. *For all $t \in [t_k, t_{k+1}]_{\mathbb{Z}}$, and $\forall k \in \mathbb{N}_0$, if $u(t) = u(t|t_k)$ then $V(x(t+1)) \le \bar{V}(t+1|t_k)$.*

Proof. For any $t \in [t_k, t_{k+1}]_{\mathbb{Z}}$, and $\forall k \in \mathbb{N}_0$, if $u(t) = u(t|t_k)$, then we can say that

$$V(x(t+1)) = x^\top(t+1)Px(t+1)$$
$$= [Ax(t)+Bu(t|t_k)+d(t)]^\top P[Ax(t)+Bu(t|t_k)+d(t)].$$

Simplifying this, and by using Assumption **(A1)**, we obtain

$$\begin{aligned} V(x(t+1)) = \; & [Ax(t)+Bu(t|t_k)]^\top P[Ax(t)+Bu(t|t_k)]+ \\ & d^\top(t)Pd(t) + 2d^\top(t)P[Ax(t)+Bu(t|t_k)] \\ = \; & [Ax(t)+Bu(t|t_k)]^\top P[Ax(t)+Bu(t|t_k)]+ \\ & \lambda_{\max}(P)(\|d(t)\|^2 + 2\|d(t)\|\,\|Ax(t)+Bu(t|t_k)\|) \\ \le \; & \bar{V}(t+1|t_k). \end{aligned}$$

□

Note that as per (2), $u(t) = u(t|t_k)$, for all $t \in [t_k, t_{k+1})_{\mathbb{Z}}$, and $\forall k \in \mathbb{N}_0$. However, in Lemma 7, we consider the hypothetical scenario $u(t) = u(t|t_k)$ for $t = t_{k+1}$. This is because at t_{k+1}, in order to first decide if a replanning of the control trajectory is required at t_{k+1}, we need to evaluate the usefulness and the likely effect of the previously computed input, $u(t|t_k)$.

Now, we present the event-triggering rule below

$$t_{k+1} = \min\left\{t > t_k : \bar{V}(t+1|t_k) > H(t,t_k)\right\}, \tag{6}$$
$$H(t,t_k) := \max\left\{\varepsilon^2, \beta^{t-t_k+1}V(x(t_k))\right\}, \tag{7}$$

where $t_0 = 0$ and $\varepsilon := \frac{D}{\sigma}$. Here $\beta \in (0,1)$ and $\sigma > 0$ are design parameters.

In summary, the closed loop system, $\mathcal{S}$, is the combination of the system dynamics (1), the parameterized control law (2), with coefficients chosen by solving (4), which are updated at the events determined by the event-triggering rule (6). That is,

$$\mathcal{S}: \; (1),(2),(4),(6). \tag{8}$$

IV. Analysis of the Event-Triggered Controller

This section analyzes the proposed event-triggered parameterized controller. First, we present a lemma that helps to prove the main result of this paper.

Lemma 8. *Consider the closed loop system (8). If $0 < \alpha < \beta < 1$ and $\sigma \le \bar{\sigma}$ where*

$$\bar{\sigma} := \min_{\tau \in [1,M]_{\mathbb{Z}}} \left\{ \frac{-\sqrt{\frac{\alpha^\tau}{\lambda_{\min}(P)}} + \sqrt{\frac{\alpha^\tau}{\lambda_{\min}(P)} + \frac{\beta^\tau - \alpha^\tau}{\lambda_{\max}(P)}}}{1 + \|A\|\bar{A}(\tau-1)} \right\},$$

$\bar{A}(\tau) := \left\|\sum_{j=0}^{\tau-1} A^j\right\|$ with $\bar{A}(0) = 0$, and $M \in \mathbb{N}$ is same as in (3), then the following statements are true.

- *If $V(x(t_k)) \ge \varepsilon^2$, for some $k \in \mathbb{N}_0$, then $\bar{V}(t_k+\tau|t_k) \le \beta^\tau V(x(t_k))$, $\forall \tau \in [1,M]_{\mathbb{Z}}$.*
- *If $V(x(t_k)) \le \varepsilon^2$, for some $k \in \mathbb{N}_0$, then $\bar{V}(t_k+\tau|t_k) \le \varepsilon^2$, $\forall \tau \in [1,M]_{\mathbb{Z}}$.*

Proof. First, consider the function

$$\gamma(\tau) := \left(\alpha^\tau + \lambda_{\max}(P)\left(1+\|A\|\bar{A}(\tau-1)\right)^2\sigma^2\right) + 2\lambda_{\max}(P)\left(1+\|A\|\bar{A}(\tau-1)\right)\sqrt{\frac{\alpha^\tau}{\lambda_{\min}(P)}}\sigma.$$

Then $\sigma \le \bar{\sigma}$ and the definition of $\bar{\sigma}$ imply that

$$\gamma(\tau) \le \beta^\tau, \quad \forall \tau \in [1,M]_{\mathbb{Z}}. \tag{9}$$

The definition of $\bar{V}(t+1|t_k)$ can be rewritten as follows,

$$\begin{aligned} \bar{V}(t+1|t_k) = & [\hat{x}(t+1|t_k)+Ae(t|t_k)]^\top P[\hat{x}(t+1|t_k)+Ae(t|t_k)] \\ & + \lambda_{\max}(P)(D^2 + 2D\|\hat{x}(t+1|t_k)+Ae(t|t_k)\|) \end{aligned}$$

where $e(t|t_k) := x(t) - \hat{x}(t|t_k)$ and $\hat{x}(t|t_k)$ is the nominal state trajectory which follows the dynamics

$$\hat{x}(t+1|t_k) = A\hat{x}(t|t_k) + Bu(t|t_k), \;\; \hat{x}(t_k|t_k) = x(t_k).$$

Then, $\forall \tau \in [1,M]_{\mathbb{Z}}$,

$$\begin{aligned} & \bar{V}(t_k+\tau|t_k) = \\ & V(\hat{x}(t_k+\tau|t_k)) + 2\hat{x}^\top(t_k+\tau|t_k)PAe(t_k+\tau-1|t_k)+ \\ & e^\top(t_k+\tau-1|t_k)A^\top PAe(t_k+\tau-1|t_k)+ \\ & \lambda_{\max}(P)(D^2 + 2D\|\hat{x}(t_k+\tau|t_k)+Ae(t_k+\tau-1|t_k)\|). \end{aligned}$$

Note that, for any $t \in [t_k, t_{k+1}]_{\mathbb{Z}}$, and $\forall k \in \mathbb{N}_0$, if $u(t) = u(t|t_k)$, then $e(t+1|t_k) = Ae(t|t_k) + d(t)$ and hence $\|e(t_k+\tau|t_k)\| \le \bar{A}(\tau)D$. By using the fact that, for any $k \in \mathbb{N}_0$ and $\forall \tau \in [0,M]_{\mathbb{Z}}$, $V(\hat{x}(t_k+\tau|t_k)) \le \alpha^\tau V(x(t_k))$ from the constraints in (3), we can say that

$$\begin{aligned} \bar{V}(t_k+\tau|t_k) \le \; & \alpha^\tau V(x(t_k)) + \lambda_{\max}(P)\left(\|A\|\bar{A}(\tau-1)D\right)^2 + \\ & 2\lambda_{\max}(P)\sqrt{\frac{\alpha^\tau V(x(t_k))}{\lambda_{\min}(P)}}\|A\|\bar{A}(\tau-1)D \; + \\ & \lambda_{\max}(P)\left[D^2 + 2D\left(\sqrt{\frac{\alpha^\tau V(x(t_k))}{\lambda_{\min}(P)}} + \|A\|\bar{A}(\tau-1)D\right)\right]. \end{aligned}$$

This implies that,

$$\begin{aligned} \bar{V}(t_k+\tau|t_k) \le & \alpha^\tau V(x(t_k)) + \lambda_{\max}(P)D^2\left(1+\|A\|\bar{A}(\tau-1)\right)^2 + \\ & 2\lambda_{\max}(P)D\left(1+\|A\|\bar{A}(\tau-1)\right)\sqrt{\frac{\alpha^\tau V(x(t_k))}{\lambda_{\min}(P)}}. \end{aligned} \tag{10}$$

Note that, in the first statement of this lemma, as $D^2 \le \sigma^2 V(x(t_k))$ we can say from (9) that

$$\bar{V}(t_k+\tau|t_k) \le \gamma(\tau)V(x(t_k)) \le \beta^\tau V(x(t_k)), \quad \forall \tau \in [1,M]_{\mathbb{Z}}.$$

This completes the proof of the first statement of this lemma.

Next note that, in the second statement of this lemma, as $V(x(t_k)) \le \frac{D^2}{\sigma^2}$, we can say from (10) that

$$\bar{V}(t_k+\tau|t_k) \le \gamma(\tau)\frac{D^2}{\sigma^2} \le \beta^\tau\frac{D^2}{\sigma^2} \le \frac{D^2}{\sigma^2}, \quad \forall \tau \in [1,M]_{\mathbb{Z}}.$$

where the last inequality follows from the fact that $\beta^\tau < 1$, $\forall \tau \in [1,M]_{\mathbb{Z}}$. This completes the proof of the second statement of this lemma. □

Next, we present the main theorem of this paper.

Theorem 9. *(Lower bound on inter-event times and global uniform ultimate boundedness of trajectories). Consider the closed loop system (8). Let $M \ge 1$ in (3) and let the conditions of Lemma 8 be satisfied. Then,*

- *The inter-event times $t_{k+1} - t_k \geq M$, $\forall k \in \mathbb{N}_0$ and if $M \geq 2$ then the inter-event times are non-trivial, i.e., $t_{k+1} - t_k > 1$, $\forall k \in \mathbb{N}_0$.*
- *If $V(x(t_k)) \leq \varepsilon^2$ for some $k \in \mathbb{N}_0$, then $V(x(t)) \leq \varepsilon^2$, $\forall t \in [t_k, \infty)_{\mathbb{Z}}$.*
- *If $V(x(t_0)) > \varepsilon^2$, then there exists a $k \in \mathbb{N}$ such that $V(x(t_k)) \leq \varepsilon^2$.*
- *The trajectories of the closed loop system (8) are globally uniformly ultimately bounded with ε^2 being the ultimate bound on $V(x)$.*

Proof. Let us prove the first statement of this theorem. Note that, according to the event-triggering rule (6), an event is triggered at $t > t_k$ if and only if $\bar{V}(t+1|t_k) > \varepsilon^2$ and $\bar{V}(t+1|t_k) > \beta^{t-t_k+1} V(x(t_k))$. Lemma 8 shows that, for any $t \in [t_k, t_k + M - 1]_{\mathbb{Z}}$, at least one of the two conditions given above is not satisfied. This implies that $t_{k+1} - t_k \geq M$ for $\forall k \in \mathbb{N}_0$. This completes the proof of the first statement.

Now, let us prove the second statement by contradiction. Let there exist $\bar{t} \in [t_k + 1, \infty)_{\mathbb{Z}}$ such that $V(x(\bar{t})) > \varepsilon^2$ and $V(x(t)) \leq \varepsilon^2$ for all $t \in [t_k, \bar{t} - 1]_{\mathbb{Z}}$. Let $t_q \geq t_k$ be such that $\bar{t} \in [t_q, t_{q+1}]$. Then, by Lemma 7, $\bar{V}(\bar{t}|t_q) \geq V(x(\bar{t})) > \varepsilon^2$. This implies, according to the event-triggering rule (6), that an event must be triggered at $t = \bar{t} - 1$, i.e., $t_{q+1} = \bar{t} - 1$, i.e., $\bar{t} = t_{q+1} + 1 \notin [t_q, t_{q+1}]_{\mathbb{Z}}$. Similarly, by using the second statement of Lemma (8), we can say that $V(x(\bar{t})) \leq \bar{V}(\bar{t}|t_{q+1}) = \bar{V}(t_{q+1} + 1|t_{q+1}) \leq \varepsilon^2$, which is a contradiction. Thus, there does not exist such a $\bar{t}$ and this completes the proof of the second statement.

Next, we prove the third statement. As $M \geq 1$, according to the event-triggering rule (6) and Lemma 7 and Lemma 8, if $V(x(t_k)) > \varepsilon^2$ for any $k \in \mathbb{N}_0$, then $V(x(t_{k+1})) \leq \max\{\beta^M V(x(t_k)), \varepsilon^2\}$. Thus $\{V(x(t_k))\}$ is a monotonically decreasing sequence, with a uniform bound $\beta^M < 1$ on the rate of decrease, as long as $V(x(t_{k+1})) > \varepsilon^2$. Hence, there must exists a $q \in \mathbb{N}$ such that $V(x(t_q)) \leq \varepsilon^2$.

Now, by using the second and the third statements, we can say that for any initial state $x(t_0)$ there exists a $T \in \mathbb{N}_0$ such that $V(x(t)) \leq \varepsilon^2$ for all $t \in [T, \infty)_{\mathbb{Z}}$. This completes the proof of this theorem. $\square$

V. NUMERICAL EXAMPLES

This section presents a numerical example to illustrate the theoretical results.

Example 1: Consider the system,

$$x(t+1) = \begin{bmatrix} 0.7 & -0.1 & -0.1 \\ 0 & 0.8 & -0.4 \\ 0 & 0 & 1.2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + d(t),$$

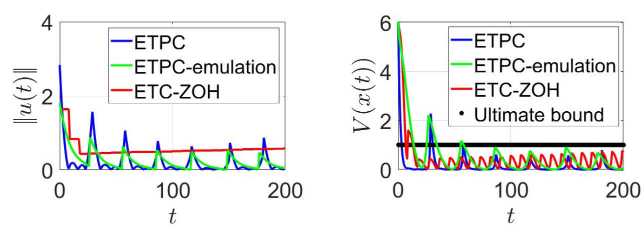
for all $t \in \mathbb{N}_0$. In this example, we consider the control input as a linear combination of the set of functions $\{1, \tau, \tau^2, \dots, \tau^p\}$. That is each control input to the plant is a polynomial of degree p. We consider the external disturbance $d(t) = \frac{0.01}{\sqrt{3}} \begin{bmatrix} \sin(50t) & \sin(20t) & \sin(10t) \end{bmatrix}^\top$ that satisfies Assumption **(A1)** with $D = 0.01$. We choose the quadratic Lyapunov function $V(x) := x^\top P x$, where $P \succ 0$ is chosen such that it satisfies the Lyapunov equation $(A + BK)^\top P (A + BK) - P = -Q$, with $Q = 0.01\mathbb{I}$ where $\mathbb{I}$ is a 3×3 identity matrix and $K = \begin{bmatrix} 0 & 0 & -0.3 \end{bmatrix}$. According to Proposition 5 and Remark 6, we can verify that the optimization problem (4) has a feasible solution for any $M \in [1, 8]_{\mathbb{Z}}$. We choose the design parameters $M = 2$, $R = 1$, $\alpha = 0.952$, $\beta = 0.99$, and $\sigma = 0.01$ which satisfy the conditions given in Lemma 8.

We compare the performance of the proposed CLF based ETPC method (ETPC-CLF) with the emulation based ETPC method (ETPC-emulation) proposed in our previous work [1] and with the typical ZOH based event-triggered control method (ETC-ZOH). In the emulation based ETPC method, we consider the same parameterized control law (2) and the event-triggering rule (6). However, at each triggering instant, the coefficients of the parameterized control law are updated by solving the following optimization problem.

$$\begin{aligned} a(k) \in \underset{a \in \mathbb{R}^{m(p+1)}}{\arg\min} \ & \sum_{t=t_k}^{t_k+N} \|u(t) - K\hat{x}(t)\|^2, \\ \text{s.t.} \ & \hat{x}(t+1) = (A+BK)\hat{x}(t), \ \hat{x}(t_k) = x(t_k), \\ & u(t) = \mathbb{P}(t - t_k)a, \ \forall t \in [t_k, t_k + N]_{\mathbb{Z}}, \\ & \mathbb{P}(0)a = Kx(t_k). \end{aligned} \tag{11}$$

In ETC-ZOH method, the control input to the plant is held constant between two successive communication time instants, i.e., $u(t) = u_k$, $\forall t \in [t_k, t_{k+1})_{\mathbb{Z}}$. We use the same event-triggering rule (6) to determine the sequence of communication time instants and at each communication time instant the control input to the plant is updated by solving the following optimization problem,

$$\begin{aligned} u_k \in \underset{u \in \mathbb{R}^m}{\arg\min} \ & \sum_{t=t_k}^{t_k+N} \left[V(\hat{x}(t)) + u^T R u \right], \\ \text{s.t.} \ & \hat{x}(t+1) = A\hat{x}(t) + Bu, \ \forall t \in [t_k, t_k + N]_{\mathbb{Z}}, \\ & \hat{x}(t_k) = x(t_k), \\ & V(\hat{x}(t)) \leq \alpha^{t-t_k} V(\hat{x}(t_k)), \ \forall t \in [t_k, t_k + M]_{\mathbb{Z}}. \end{aligned} \tag{12}$$



(a) Control input (b) Evolution of $V(x(t))$

Fig. 2: Simulation results of Example 1.

Figure 2 presents the simulation results with $p = 3$, $N = 25$ and $x(0) = \begin{bmatrix} 2 & 5 & 6 \end{bmatrix}^\top$. Figure 2a presents the evolution of norm of $u(t)$ and it shows that the proposed ETPC-CLF method offers smaller values for $\|u(t)\|$ at most of the time instants compared to the other two methods. Figure 2b presents the evolution of $V(x)$ along the system trajectory and it shows that $V(x)$ converges to the ultimate bound $\varepsilon^2 = 1$ in all the three cases. Even though $u(t)$ and $V(x(t))$ are discrete-time signals, for ease of visualization, we plot them as continuous-time signals.

Next, we consider 100 initial conditions uniformly sampled from a sphere with a specific radius and we calculate the average inter-event time (AIET) and the minimum inter-event time (MIET) over 100 events for each initial condition with $N = 30$, and $p = 3$. These observations are tabulated in Table I. Note that, for the given choice of control law and

TABLE I: Average of AIET and minimum of MIET, over a set of initial conditions, for ETPC-CLF, ETPC-emulation and ETC-ZOH with $N = 30$ and $p = 3$.

	Average of AIET	**Minimum of MIET**
ETPC-CLF	35.2348	32
ETPC-emulation	25.7287	25
ETC-ZOH	9.2121	2

the event-triggering rule, the proposed ETPC-CLF method performs better, in terms of the AIET and MIET, compared to the ETC-ZOH method and ETPC-emulation based method. Note also that, in the ETPC-CLF method, both the AIET and the MIET are greater than N. This shows that the proposed method performs better, in terms of the AIET and MIET, compared to the event-triggered model predictive control method [13], [14] in which the maximum inter-event time is typically chosen as the prediction horizon length N.

We repeat the procedure for the proposed ETPC-CLF method for different values of N and p, and the observations are tabulated in Table II. In Table II, we can see that there is an increasing trend in the values of AIET and MIET as N or p increases. Note that as N increases, the finite horizon length of the optimization problem (3) increases and hence leads to larger inter-event time. This is an advantage compared to the ETPC-emulation method proposed in [1] where there is a decreasing trend in inter-event times as N increases. Also that choosing a larger p helps to choose a control input from a larger input space and hence leads to better performance.

TABLE II: Average of AIET and minimum of MIET, over a set of initial conditions, for ETPC-CLF for different values of N and p.

	N					
	10		20		30	
p	**AIET**	**MIET**	**AIET**	**MIET**	**AIET**	**MIET**
2	14.1268	13	23.6609	23	33.1982	31
3	15.2476	15	24.6605	23	35.2348	32
4	16.0283	16	25.3624	25	36.62655	33

VI. CONCLUSION

In this paper, we proposed an event-triggered parameterized control method using a control Lyapunov function approach for discrete time linear systems with external disturbances. We designed a parameterized control law and an event-triggering rule that guarantee global uniform ultimate boundedness of the trajectories of the closed loop system and non-trivial inter-event times. We illustrated our results through numerical examples. We observed that for the given choice of control law and event-triggering rule, the proposed control method performs better in terms of the AIET and the MIET compared to other existing methods such as emulation based ETPC, ZOH based ETC and event-triggered MPC.

REFERENCES

[1] A. Rajan and P. Tallapragada, "Event-triggered parameterized control for stabilization of linear systems," *Accepted at 62nd IEEE Conference on Decision and Control (CDC)*, 2023.

[2] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, 2007.

[3] W. P. M. H. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *IEEE Conference on Decision and Control (CDC)*, 2012, pp. 3270–3285.

[4] M. Lemmon, "Event-triggered feedback in control, estimation, and optimization," in *Networked control systems*. Springer, 2010, pp. 293–358.

[5] D. Tolić and S. Hirche, *Networked control systems with intermittent feedback*. CRC Press, 2017.

[6] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2030–2042, 2010.

[7] W. P. M. H. Heemels, M. C. F. Donkers, and A. R. Teel, "Periodic event-triggered control for linear systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 847–861, 2013.

[8] E. García and P. Antsaklis, "Model-based event-triggered control for systems with quantization and time-varying network delays," *IEEE Transactions on Automatic Control*, vol. 58, pp. 422–434, 02 2013.

[9] W. Heemels and M. Donkers, "Model-based periodic event-triggered control for linear systems," *Automatica*, vol. 49, no. 3, pp. 698–711, 2013.

[10] H. Zhang, D. Yue, X. Yin, and J. Chen, "Adaptive model-based event-triggered control of networked control system with external disturbance," *IET Control Theory & Applications*, vol. 10, no. 15, pp. 1956–1962, 2016.

[11] Z. Chen, B. Niu, X. Zhao, L. Zhang, and N. Xu, "Model-based adaptive event-triggered control of nonlinear continuous-time systems," *Applied Mathematics and Computation*, vol. 408, p. 126330, 2021.

[12] L. Zhang, J. Sun, and Q. Yang, "Distributed model-based event-triggered leader–follower consensus control for linear continuous-time multiagent systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 10, pp. 6457–6465, 2021.

[13] H. Li and Y. Shi, "Event-triggered robust model predictive control of continuous-time nonlinear systems," *Automatica*, vol. 50, no. 5, pp. 1507–1513, 2014.

[14] F. D. Brunner, W. Heemels, and F. Allgower, "Robust event-triggered MPC with guaranteed asymptotic bound and average sampling rate," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, p. 5694 – 5709, 2017.

[15] H. Li, W. Yan, and Y. Shi, "Triggering and control codesign in self-triggered model predictive control of constrained systems: With guaranteed performance," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 4008–4015, 2018.

[16] A. Li and J. Sun, "Resource limited event-triggered model predictive control for continuous-time nonlinear systems based on first-order hold," *Nonlinear Analysis: Hybrid Systems*, vol. 47, p. 101273, 2023.

[17] K. Hashimoto, S. Adachi, and D. V. Dimarogonas, "Self-triggered model predictive control for nonlinear input-affine dynamical systems via adaptive control samples selection," *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 177–189, 2017.

[18] B. Demirel, V. Gupta, D. E. Quevedo, and M. Johansson, "On the trade-off between communication and control cost in event-triggered dead-beat control," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 2973–2980, 2017.

[19] A. Rajan and P. Tallapragada, "Event-triggered parameterized control of nonlinear systems," *IEEE Control Systems Letters*, vol. 8, pp. 1673–1678, 2024.

[20] H. V, A. Rajan, B. Amrutur, and P. Tallapragada, "Event-triggered polynomial control for trajectory tracking by unicycle robots," 2025. [Online]. Available: https://arxiv.org/abs/2308.15834

[21] A. Rajan, A. Kattepur, and P. Tallapragada, "Co-design of polynomial control law and communication scheduling strategy for multi-loop networked control systems," in *2024 Tenth Indian Control Conference (ICC)*, 2024, pp. 220–225.

[22] S. Das, S. Ganguly, M. Anjali, and D. Chatterjee, "Towards continuous-time mpc: A novel trajectory optimization algorithm," in *2023 62nd IEEE Conference on Decision and Control (CDC)*, 2023, pp. 3276–3281.