Distributed control of vehicle strings under finite-time and safety specifications

Pavankumar Tallapragada Jorge Cortés

Abstract—**This paper studies an optimal control problem for a string of vehicles with safety requirements and finite-time specifications on the approach time to a target region. Our problem formulation is motivated by scenarios involving autonomous vehicles circulating on arterial roads with intelligent management at traffic intersections. We propose a provably correct distributed control algorithm that ensures that the vehicles satisfy the finite-time specifications under speed limits, acceleration saturation, and safety requirements. The safety specifications are such that collisions can be avoided even in cases of communication failure. We also discuss how the proposed distributed algorithm can be integrated with an intelligent intersection manager to provide information about the feasible approach times of the vehicle string and a guaranteed bound of its time of occupancy of the intersection. Our simulation study illustrates the algorithm and its properties regarding approach time, occupancy time, and fuel and time cost.**

Index Terms—**vehicle strings, distributed control, intelligent transportation, networked vehicles, state-based intersection management**

I. Introduction

In this paper we are motivated by the vision for urban traffic with coordinated computer-controlled vehicles and networked intersection managers. Emerging technologies in autonomy and communication offer the opportunity to radically redesign our transportation systems, reducing road accidents and traffic collisions and positively impacting safety, traveling ease, travel time, and energy consumption. For example, cruise control and coordination of vehicles (e.g., platooning) could ensure smoother (with reduced stop-and-go), safer and fuel-efficient traffic flow. This vision involves, among many other things, scheduling of vehicles' usage of an intersection and the vehicles optimally meeting those schedules under safety constraints. Distributed algorithmic solutions are necessary in order to produce real-time implementations under the computationally-heavy tasks involved. To this end, here we explore a generalized problem of distributed control of vehicle strings under specifications of reaching a target in a fixed finite time while respecting safety specifications.

Literature review: The control and coordination of multi-vehicle systems in the context of transportation has a long history starting with the platooning problem formalized in [2]. This classical and active area of research is so vast that a fair and complete overview is beyond the scope of this paper. The recent survey papers [3], [4], however, provide a good introduction to the topic of vehicle platoon control and its literature. The prototypical aim in vehicle platooning is to asymptotically achieve a given constant inter-vehicular distance (or a given constant headway (time) between successive vehicles [5]) while ensuring all vehicles move at a desired speed. Hence the topic is often called 'string stability'. The problem is typically formalized as an asymptotic stabilization problem or an infinite-horizon optimal control problem, with non-collision constraints that ensure vehicles are separated by at least a given fixed amount. With some exceptions [6]–[8], constraints on acceleration control or on state variables, such as speed limits, are not considered. Recent research on string stability also seeks to address the challenges that arise due to coordination via wireless communication channels such as sampling [9], communication delays [10], [11] and limited communication range [12]. The work [13] examines the question of whether local feedback is sufficient to ensure coherence of large networks under stochastic disturbances. Given that the string stability problem has largely been motivated by cruise control of vehicle platoons on highways, it is not surprising that finite-time constraints on the states of the vehicles are also usually not considered. This is a major difference with respect to our treatment in this paper. Since we are motivated by the problem of coordination of a group of vehicles on arterial roads with intersections, the consideration of finite-time constraints on the vehicle string is key.

In the literature on coordination-based intersection management, the explicit control of platoons has been rarely considered with some notable exceptions. The works [14], [15] describe a hierarchical setup, with a central coordinator verifying and assigning reservations, and with vehicles planning their trajectories locally to platoon and to meet the assigned schedule. The proposed solution is based on multiagent simulations, an important difference with respect to our approach. In [16], a polling-systems approach is adopted to assign schedules, and then optimal trajectories for all vehicles are computed sequentially in order. Such optimal trajectory computations are costly and depend on other vehicles' detailed plans, and hence the system is not robust. In this literature too, a non-collision constraint is imposed on the vehicles. The work [17] is an exception in that it requires the minimum separation between any two consecutive vehicles to be a function of the vehicles' velocities. We call such a constraint as a *safety constraint*, which is a generalization of a non-collision requirement.

Statement of contributions: Motivated by intelligent management of traffic intersections, we formulate an optimal

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Pavankumar Tallapragada is with the Department of Electrical Engineering, Indian Institute of Science, Bengaluru pavant@ee.iisc.ernet.in and Jorge Cortés is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego cortes@ucsd.edu

control problem for a vehicular string with safety requirements and finite-time specifications on the approach time to a target region. The first contribution is the design of a distributed control algorithm for the vehicle string that ensures that the vehicles satisfy the finite-time specifications, while guaranteeing system-wide safety and subject to speed limits and acceleration saturation. Additionally, each vehicle seeks to optimally control its trajectory whenever safety is not immediately at risk. The algorithm design combines three main elements: an uncoupled controller ensuring that a vehicle arrives at the intersection at a designated time when the preceding vehicle is sufficiently far in front; a safe-following controller ensuring that the vehicle follows the preceding vehicle safely when the latter is not sufficiently far in front; and a policy to switch between these two controllers. The second contribution is the analysis of the convergence and performance properties of the vehicle string under the proposed distributed control design. We provide guarantees on vehicle safe following and the approach times to the target region. The safe-following specifications are such that each vehicle maintains at all times sufficient distance from its preceding vehicle so as to have the ability to come to a complete stop without collisions irrespective of the preceding vehicle's control action. This notion has the advantage of ensuring safety even in cases of communication failures, which may not be the case for a solution computed with only non-collision constraints. We also establish that the prescribed approach time of a vehicle can be met provided it is sufficiently far from the actual approach time of the previous vehicle in the string. If this is not the case, then we also provide an upper bound on the difference between the actual approach times of consecutive vehicles. The third contribution is the application to traffic intersection management of our distributed control design. We describe how the various constraints and parameters of the problem can be integrated with an intelligent intersection manager to provide it with information about the approach time of the first vehicle in the string and a guaranteed bound of the occupancy time of the intersection by the string. Our simulation study illustrates the results and provides insights on the algorithm executions and their dependence on the problem parameters.

Organization: The rest of the paper is organized as follows. Section II details the problem formulation and discusses its connection with intelligent intersection management. Section III presents the design of the distributed control for the vehicle string and Section IV derives convergence and performance guarantees on its executions. Section V illustrates our results in a simulation study. Finally, Section VI contains concluding remarks and our ideas for future work.

Notation: We present here some basic notation used throughout the paper. We let $\mathbb{R}$, $\mathbb{R}_{<0}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the set of real, negative real, integer, positive integer, and nonnegative integer numbers, respectively. Given $a \leq b$, $[u]_a^b$ denotes the number u lower and upper saturated by a and b respectively, i.e.,

$$[u]_a^b \triangleq \begin{cases} a, & \text{if } u \leq a, \\ u, & \text{if } u \in [a,b], \\ b, & \text{if } u \geq b. \end{cases}$$

II. Problem statement

Consider a string of vehicles on its way to a target region as in Figure 1. The vehicles are labeled $\{1,\ldots,N\}$ and all have the same length L. The line segment represents a road and the target region is the interval $[0,\Delta]$, representing an intersection. The position of the (front of the) j^{th} vehicle is x_j. We assume that initially the vehicles are yet to approach the target and hence their initial positions belong to $\mathbb{R}_{<0}$. Without loss of generality, vehicles are indexed in ascending order starting from the vehicle closest to the target region at the initial time.

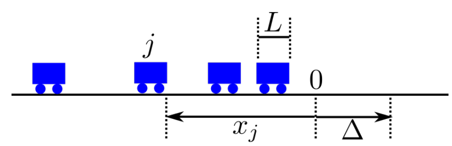


Fig. 1. A string of vehicles, each of length L, on a road to a target at 0. The position of the (front of the) j^{th} vehicle is x_j.

Constraints: The constraints on the vehicles' motion arise from their dynamics, specifications regarding their approach to the target region, and safety requirements. We describe them next. The dynamics of vehicle j is the fully actuated second-order system,

$$\dot{x}_j(t) = v_j(t), \tag{1a}$$
$$\dot{v}_j(t) = u_j(t), \tag{1b}$$

where $v_j \in \mathbb{R}$ is the velocity and $u_j(t) \in [u_m, u_M]$, with $u_m \leq 0 \leq u_M$, is the input acceleration. The vehicles must respect a maximum speed limit, v^M, imposed on the road ($v_j(t)$ must belong to $[0, v^M]$ for all $t \geq 0$).

Each vehicle j is given an exogenous *prescribed approach time* τ_j – the time at which vehicle j is to reach the beginning of the target region, i.e., the origin. Clearly, an arbitrary set of $\{\tau_j\}_{j=1}^N$ may not be feasible. We let T_j^a denote the actual *approach time* of vehicle j at target region, i.e., $x_j(T_j^a) = 0$. Further, we require that the velocity of each vehicle j at its approach time T_j^a and subsequently be at least ν^{nom}. Finally, vehicles are required to maintain a safe distance between them at all times. Specifically, the minimum separation of vehicles at any given time must be such that there always exists a control action for each vehicle to come to a stop safely even without coordination. Clearly, such a safe-following distance needs to be a function of the vehicles' velocities, which we denote by $\mathcal{D}(v_{j-1}(t), v_j(t))$ for the pair of vehicles $j-1$ and j. The formal definition of this function is postponed to the next section. Then, the safety constraint for $j \in \{2,\ldots,N\}$ is

$$x_{j-1}(t) - x_j(t) \geq \mathcal{D}(v_{j-1}(t), v_j(t)), \quad \forall t \geq 0.$$

Remark II.1. *(Safety constraints are more robust to loss of coordination than non-collision constraints).* Typically in the literature, with the exception of [17], non-collision constraints are imposed. These take the form $x_{j-1}(t) - x_j(t) \geq L$. However, non-collision constraints are not robust to loss of coordination due to communication failures or otherwise. On the other hand, the stricter safety constraints guarantee a higher degree of robustness in that there always exists a control action for each vehicle to come to a stop safely even with a loss of coordination. •

Objective: Under the constraints specified above, we seek a design solution that minimizes the cost function C modeling cumulative fuel cost,

$$C \triangleq \sum_{j\in\{1,\dots,N\}} \int_0^{T_j^{\text{exit}}} |u_j|\mathrm{d}t, \tag{2}$$

where T_j^{exit} is the time at which the vehicle j completely exits the target region, i.e., $x_j(T_j^{\text{exit}}) = \Delta + L$. This is an optimal control problem with bounds on the state and control variables and inter-vehicular safety requirements. We seek to design a strategy that allows the vehicle string to solve it in a distributed fashion. By distributed, we mean that each vehicle j receives information only from vehicle $j-1$, so that collisions can be avoided and the algorithm is implementable in real time. We envision the resulting distributed vehicular control to be a part of a larger traffic management solution.

Given the distributed and real-time requirements on the algorithmic solution, we do not insist on exact optimality and instead focus on obtaining sub-optimal solutions based on switching between an optimal control mode and a safe-following mode. In addition, we also seek to characterize conditions under which the approach time of each vehicle is equal to its prescribed approach time (i.e., $T_j^a = \tau_j$ for $j \in \{1,\dots,N\}$). Failing feasibility ($T_j^a \neq \tau_j$ for some j), we aim for our solution to minimize the intersection's *occupancy time* $T_N^{\text{exit}} - T_1^a$ of the vehicle string and seek to provide an upper bound for it.

Remark II.2. *(Connection with intelligent intersection management).* The motivation for the problem considered here is to enable control of a string of computer-controlled, networked vehicles on roads with intersections. By networked vehicles, we mean vehicles equipped with vehicle-to-vehicle (V2V) and vehicle-to-infrastructure (V2I) communication capabilities. We envision a system where vehicles communicate their state to an *intersection manager* (IM), which then prescribes a schedule for the usage of the intersection by the vehicles. In this paper, we do not address the communication and decision making aspects related to the interaction between the intersection manager and the vehicles. In the problem posed here, the target region corresponds to an intersection, the prescribed approach times are given by the IM to the vehicles, and the constraint of a minimum approach velocity ensures efficient usage of the intersection. The finite-time specifications, bounded controls, speed limits, and explicit safety constraints distinguish this work from the literature on string stability, which instead focuses on asymptotic stability or infinite horizon optimal control with only non-collision constraints [3], [4]. We see the solution to the problem formulated here as one of the many necessary building blocks towards the development of such intelligent intersection management capabilities. •

III. Design of `local vehicular` controller

In this section we design the distributed vehicle control termed `local vehicular` controller. To do this, we begin by introducing two useful notions: safe-following distance, as a way of ensuring safety at present as well as in the future, and relaxed feasible approach times ignoring safety constraints.

A. Safe-following distance

The following notion of safe-following distance plays an instrumental role in guaranteeing the inter-vehicle safety requirement in our forthcoming developments.

Definition III.1. *(Safe-following distance). The maximum braking maneuver (MBM) of a vehicle is a control action that sets its acceleration to u_m until the vehicle comes to a stop, at which point its acceleration is set to 0 thereafter. A quantity $\mathcal{D}(v_{j-1}(t), v_j(t))$ is a* safe-following distance *at time t for the consecutive vehicles $j-1$ and j if $x_{j-1}(t) - x_j(t) \geq \mathcal{D}(v_{j-1}(t), v_j(t)) \geq L$ and, if each of the two vehicles were to perform the MBM, then they would be safely separated, $x_{j-1} - L \geq x_j$, until they come to a complete stop.* •

Given the notion of safe-following distance, we ensure inter-vehicular safety by requiring for all $j \in \{2,\dots,N\}$ that

$$x_{j-1}(t) - x_j(t) \geq \mathcal{D}(v_{j-1}(t), v_j(t)), \ \forall t. \tag{3}$$

According to Definition III.1, a safe-following distance is not uniquely defined, which in fact provides a certain leeway in designing the local vehicle control. The following result identifies a specific safe-following distance.

Lemma III.2. *(Safe-following distance as a function of vehicle velocities). Let $j-1$ and j be a pair of vehicles, with j following $j-1$. Then, the continuous function $\mathcal{D}$ defined by*

$$\mathcal{D}(v_{j-1}(t), v_j(t)) = \\ L + \max\left\{0, \frac{1}{-2u_m}\left((v_j(t))^2 - (v_{j-1}(t))^2\right)\right\}, \tag{4}$$

provides a safe-following distance at time t for the pair of vehicles $j-1$ and j.

Proof. If a vehicle j with dynamics (1) were to perform the MBM at the current time t until it comes to a complete stop at $t_j^{\text{stop}} = -v_j(t)/u_m$, then

$$x_j(t_j^{\text{stop}}) = x_j(t) + \frac{(v_j(t))^2}{-2u_m}.$$

If $v_j(t) \geq v_{j-1}(t) \geq 0$, then the safe-following distance is found by setting

$$x_{j-1}(t_{j-1}^{\text{stop}}) - x_j(t_j^{\text{stop}}) \geq L.$$

If on the other hand $v_{j-1}(t) \geq v_j(t) \geq 0$, then the vehicles are in fact closest at time t and the condition $x_{j-1}(t) - x_j(t) \geq L$ is sufficient to ensure subsequent safety. Hence (4) provides a safe following distance. □

The safe-following distance function $\mathcal{D}$ defined in (4) has the following useful monotonicity properties: if the first argument is fixed, then the function is monotonically non-decreasing; instead, if the second argument is fixed, then the function is monotonically non-increasing.

Remark III.3. *(Intra-branch safety under communication failure).* Note that the safety constraint (3) for each pair of consecutive vehicles on the same branch is more than just non-collision constraint. The safety constraints always ensure, for each vehicle, the existence of a control action that can safely bring the vehicle to a complete stop irrespective of the actions of the vehicle preceding it. Thus, in particular, if communication were to break down between any pair of consecutive vehicles and if the communication failure were detected then the following vehicle can safely come to a stop. •

B. Feasible approach times ignoring safety constraints

Here we provide bounds on the feasible approach times by ignoring the safety constraints. We rely on these bounds in our controller design later to ensure the existence of an optimal controller that guarantees a vehicle arrives at the intersection at a designated time when the preceding vehicle is sufficiently far in front.

We start by defining the earliest and latest times of approach of each vehicle at the target region, ignoring other vehicles. Formally, let τ_j^e be the earliest time vehicle j can reach the target region while ignoring the safety constraints (3). This time can be computed by considering the trajectory with the initial condition $(x_j(0), v_j(0))$ and the control policy with maximum acceleration ($u_j = u_M$) until $v_j(t) = v_M$ and zero acceleration thereafter. It can be easily verified that $\tau_j^e = \mathcal{T}(-x_j(0), v_j(0))$, where

$$\mathcal{T}(d, v) \triangleq \begin{cases} \frac{\sqrt{2u_M d + v^2} - v}{u_M}, & 2u_M d \leq (v^M)^2 - v^2, \\ \frac{v^M - v}{u_M} + \frac{2u_M d - (v^M)^2 + v^2}{2u_M v^M}, & 2u_M d \geq (v^M)^2 - v^2. \end{cases} \quad (5)$$

Similarly, let τ_j^l be the latest time vehicle j can reach the target region ignoring the safety constraints by considering trajectories with maximum deceleration. Note that this could possibly result in $\tau_i^l = \infty$.

An important observation is that there might not exist any approach time T_j^a during the interval $[\tau_j^e, \tau_j^l]$ such that the minimum approach velocity constraint $v_j(T_j^a) \geq \nu^{\text{nom}}$ is satisfied (even when safety constraints are ignored). The following result presents a sufficient condition that guarantees the existence of such approach times even when safety constraints are considered.

Lemma III.4. *(Existence of a feasible approach time). Suppose the initial position of vehicle $j \in \{1, \ldots, N\}$ satisfies*

$$x_j(0) \leq \frac{(v^M)^2}{2u_m} - \frac{(\nu^{nom})^2}{2u_M}, \quad (6)$$

then $\tau_j^l = \infty$, and for any $\tau \in [\tau_j^e, \infty)$ there exists a control action that, ignoring the safety constraints (3), ensures that $x_j(\tau) = 0$ and $v_j(\tau) \geq \nu^{nom}$. Furthermore, if the safety constraints (3) are satisfied initially at time 0, then the set of feasible approach times $[\bar{\tau}_j^e, \infty)$ is non-empty and $\bar{\tau}_j^e \geq \tau_j^e$ for all $j \in \{1, \ldots, N\}$.

Proof. The condition on $x_j(0)$ implies that vehicle j can come to a complete stop, wait for an arbitrarily long time and then accelerate to a speed of at least ν^{nom} before arriving at the beginning of the target region so that $\tau_j^l = \infty$. Further, it also means that $v(\tau_j^e) \geq \nu^{\text{nom}}$ under the control action used for computing τ_j^e. Thus, for any $\tau \in [\tau_j^e, \infty)$ there exists a control action that, ignoring the safety constraints (3), ensures that $x_j(\tau) = 0$ and $v_j(\tau) \geq \nu^{\text{nom}}$.

If in addition, safety constraints (3) are satisfied initially, then existence of a feasible approach time is guaranteed because vehicle j can safely decelerate at the maximum rate until it comes to a complete stop while ensuring safety with vehicles $j-1$ and $j+1$, then wait for enough time to avoid collision with vehicle $j-1$ and accelerate back to ν^{nom} before reaching the target region at T_j^a. Finally, if $T_j^a = t_1$ is feasible, then so is $T_j^a = t_2$ for all $t_2 \geq t_1$ by increasing the deceleration time or the wait time. □

Note that $[\bar{\tau}_j^e, \infty)$, the actual set of feasible approach times for vehicle j, depends on all the constraints, the initial conditions and τ_i for all the vehicles $i \in \{i, \ldots, N\}$. As a result, it is not readily computable. In contrast, the relaxed bound $[\tau_j^e, \infty)$ is easy to compute and depends only on the data related to vehicle j, which makes it useful in the control design procedure.

In the rest of the paper we assume that conditions of Lemma III.4 are satisfied for all vehicles $j \in \{1, \ldots, N\}$ (so that feasible approach times satisfying the minimum velocity requirement are guaranteed to exist) and that $\tau_j \in [\tau_j^e, \infty)$, for all $j \in \{1, \ldots, N\}$.

C. Controller design

Here we introduce our distributed controller design, which is composed by three main elements: (i) an uncoupled controller ensuring that the vehicle arrives at the intersection at a designated time if the presence of all other vehicles is ignored. This controller is applied when the preceding vehicle is sufficiently far in front, (ii) a safe-following controller ensuring that the vehicle follows the preceding vehicle safely when the latter is not sufficiently far in front; and (iii) a rule to switch between the two controllers.

1) Uncoupled controller: We let

$$(t, x_j, v_j) \mapsto g_{uc}(\tau_j, t, x_j, v_j)$$

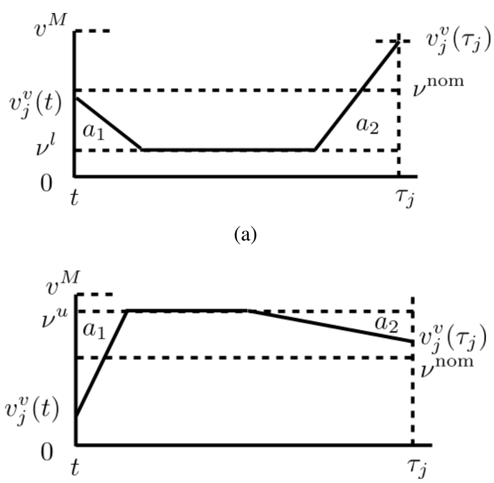
be a feedback controller that ensures $x_j(\tau_j) = 0$ for the dynamics (1) starting from the current state $(x_j(t), v_j(t))$ at time t, respecting the control and velocity constraints, but not necessarily the inter-vehicle safety constraints. We refer to it as the *uncoupled* controller. Here, we let g_{uc} be the optimal feedback controller that is obtained in the sense of receding horizon control (RHC) [18]. The cost function that is minimized in an open-loop manner as part of RHC is

$$\int_t^{\tau_j} |u_j(s)| ds.$$

For simplicity we restrict the candidate velocity profiles, each of which determines uniquely a control trajectory u_j, to the two classes shown in Figure 2. Thus, the optimization variables in RHC are a_1, a_2 (the areas of the indicated triangles), $v_j(\tau_j)$, ν^l and ν^u. The constraints are $\nu^l \in [0, v_j(t)]$, $\nu^u \in$

$[v_j(t), v^M]$, $v_j(\tau_j) \in [\nu^{\text{nom}}, v^M]$, $a_1, a_2 \geq 0$ and that the total area under the curve (corresponding to the distance traveled) is equal to $-x_j(t)$.

Remark III.5. *(Alternative implementation of the uncoupled feedback controller).* In contrast to the receding horizon approach, the uncoupled controller g_{uc} may also be implemented as a feedback controller. This may be done by analytically solving off-line the initial optimal control action as a function of τ_j, t, x_j, v_j. In our particular problem, due to the specific structure of the candidate solutions, shown in Figure 2, this analytical computation could be done exhaustively. Such computations would result in a g_{uc} that is a switched controller (with analytical expressions) as a function of τ_j, t, x_j, v_j. Further, as a vehicle's state (x_j, v_j) and time t evolve continuously with time t, there are only a few modes to which g_{uc} could switch to at any given time. Thus, in practice, such a switched feedback controller g_{uc} could be implemented quite efficiently. •



(a)

(b)

Fig. 2. (a) and (b) show the two classes of open-loop candidate velocity profiles. The optimal profile takes one of the two forms depending on the velocity $v_j(t)$, ν^{nom}, v^M, τ_j and the distance to go $-x_j(t)$. The uncoupled feedback controller g_{uc} may be obtained by executing the open-loop optimal control in a receding horizon manner.

Remark III.6. *(Optimality of the controller).* Assuming there exists a feasible controller that ensures the vehicle j approaches the intersection at τ_j with a minimum velocity of ν^{nom}, ignoring the safety constraints and given the current time t and the vehicle state $(x_j(t), v_j(t))$, then there exists an optimal solution with piecewise-constant-rate velocity profiles as shown in Figure 2. We can see this statement to be true by observing that in a given time $\tau_j - t$, the minimum and maximum travel distances are obtained with velocity profiles belonging to the family depicted in Figure 2, and that every other intermediate travel distance is obtained by a continuous variation of the velocity profiles within the family. •

It is worth noticing that the control g_{uc} assumes the presence of no other vehicles. Thus, the actual approach time, T_j^a, of the vehicle j may be later than τ_j. Note that T_j^a could not be earlier than τ_j because each vehicle j receives information only from vehicle $j-1$ and it never attempts to approach the intersection before τ_j.

Note that a feasible g_{uc} exists for each vehicle at $t = 0$. This is because under the conditions of Lemma III.4, the feasible approach times ignoring the safety constraints is $[\tau_j^e, \infty)$ and we assume that $\tau_j \in [\tau_j^e, \infty)$. However, at a future time t, such a feasible g_{uc} might not exist because the vehicle is slowed down by preceding vehicles and no control exists to ensure $T_j^a = \tau_j$ along with the other constraints. Additionally, for $t > T_j^a$, i.e., after the vehicle enters the target region, the optimal controller is not well defined and does not exist. As a shorthand notation, we use $\exists \mathcal{F}_j$ (respectively $\nexists \mathcal{F}_j$) to denote the existence (respectively, lack thereof) of an optimal uncoupled control. In order for the control g_{uc} to be well defined at all times and for all states, we extend it as

$$g_{uc}(\tau_j, t, x_j, v_j) \triangleq u_M, \quad \text{if } \nexists \mathcal{F}_j.$$

2) Controller for safe following: As mentioned earlier, this controller is applied only when a vehicle is sufficiently close to the vehicle preceding it. Besides maintaining a safe-following distance, the controller must also ensure that the resulting evolution of the vehicles does not result in undue delays in approach times. Here, we present a design to achieve these goals. For a pair of vehicles $j-1$ and j, we define the *safety ratio* as

$$\sigma_j(t) \triangleq \frac{x_{j-1}(t) - x_j(t)}{\mathcal{D}(v_{j-1}(t), v_j(t))}, \tag{7}$$

which is the ratio of the actual inter-vehicle distance to the safe-following distance. Hence, the requirement (3) can be equivalently expressed as stating that $\sigma_j(t)$ should remain above 1 at all times. Notice from the definition (4) of the safe-following distance that, if $v_{j-1}(t) > v_j(t)$, then $\sigma_j(t)$ increases at time t and safety is guaranteed. Thus, it is sufficient to design a controller that ensures safe following when $v_j(t) \geq v_{j-1}(t)$. For vehicle j, we denote $\zeta_j \triangleq (v_{j-1}, v_j, \sigma_j)$. Define the *unsaturated* controller g_{us} by

$$g_{us}(\zeta_j, u_{j-1}) \triangleq \begin{cases} u_{j-1}, & \text{if } v_j = 0, \\ \left(\frac{v_{j-1}}{v_j}\left(1 + \sigma_j \frac{u_{j-1}}{-u_m}\right) - 1\right)\left(\frac{-u_m}{\sigma_j}\right), & \text{if } v_j > 0. \end{cases}$$

The rationale behind this definition is as follows. Since it is sufficient to design a controller that ensures safe following when $v_j(t) \geq v_{j-1}(t)$, if $v_j = 0$, then we need to consider only the case $v_{j-1} = 0$. In this case, the definition of g_{us} ensures that the vehicle j stays at rest as long as vehicle $j-1$ is at rest, and starts moving only when $j-1$ starts moving again. Further, since the relative velocity and acceleration in this case would be zero, we see that σ_j stays constant. On the other hand, if the vehicle is moving, $v_j > 0$, then g_{uc} is designed to make sure that σ_j stays constant (we show this formally in Lemma IV.1), thus ensuring safety. However, in this latter case, g_{us} might cause v_j to exceed v^M. Further, we would like the vehicle to continue using the optimal uncoupled controller if it does not affect the safety by decreasing σ_j. These considerations motivate our definition of the *safe-following* controller as

$$g_{sf}(t,\zeta_j,u_{j-1}) \triangleq \min\{g_{uc}(\tau_j,t,x_j,v_j), g_{us}(\zeta_j,u_{j-1})\}. \quad (8)$$

3) `local vehicular` *controller:* Here, we design the local vehicle controller by specifying a rule to switch between the uncoupled controller g_{uc} and the safe-following controller g_{sf}. To make precise whether two vehicles are sufficiently far from each other, we introduce the *coupling set* $\mathcal{C}_s$ defined by

$$\mathcal{C}_s \triangleq \{(\mathrm{v}_1,\mathrm{v}_2,\sigma) : \mathrm{v}_2 \geq \mathrm{v}_1 \text{ and } \sigma \in [1,\sigma_0]\}, \quad (9)$$

where $\sigma_0 > 1$ is a design parameter. The value of this parameter marks when the safety ratio is considered to be sufficiently close to 1 that action is required to prevent inter-vehicle collision. This criteria is more conservative (resp. aggressive) the further (resp. closer) σ_0 is to 1. If $\zeta_j \in \mathcal{C}_s$, then vehicle j is going at least as fast as the vehicle in front of it, and their safety ratio is sufficiently close to 1 that action is required. With this in mind, we define the `local vehicular` controller for vehicle j, to make sure it uses the safe-following controller when it is in the coupling set, and the uncoupled controller otherwise. Formally,

$$u_j(t) = \begin{cases} g_{uc}, & \text{if } \zeta_j \notin \mathcal{C}_s,\ v_j < v^M, \\ [g_{uc}]_{u_m}^0, & \text{if } \zeta_j \notin \mathcal{C}_s,\ v_j = v^M, \\ g_{sf}, & \text{if } \zeta_j \in \mathcal{C}_s,\ v_j < v^M, \\ [g_{sf}]_{u_m}^0, & \text{if } \zeta_j \in \mathcal{C}_s,\ v_j = v^M. \end{cases} \quad (10)$$

Note that $[g_{uc}]_{u_m}^0 \neq g_{uc}$ only if $\nexists \mathcal{F}_j$.

IV. Evolution of the vehicle string

In this section, we analyze the evolution of the vehicle string under the distributed controller designed in Section III-C. Specifically, we characterize to what extent the controller allows the vehicles to meet the specifications on the vehicle string regarding safety and approach to the target region.

A. Vehicle behavior under safe following

Here, we study the dynamical behavior of the vehicles when they are in the coupling set, i.e., when they operate under the safe-following controller g_{sf} in (8). The next result identifies conditions under which the safety ratio remains constant and the unsaturated controller exceeds the maximum acceleration.

Lemma IV.1. *(Vehicle behavior in the coupling set). For $j \in \{2,\dots,N\}$, let $t \in \mathbb{R}_{\geq 0}$ such that $\zeta_j(t) = (v_{j-1}(t), v_j(t), \sigma_j(t)) \in \mathcal{C}_s$ and $u_{j-1}(t) \in [u_m, u_M]$. Then, the following hold:*

(a) $g_{us}(\zeta_j,u_{j-1}) \in [u_m,u_M]$,

(b) *If $v_j < v^M$ and $g_{sf}(t,\zeta_j,u_{j-1}) = g_{us}(\zeta_j,u_{j-1})$ or if $v_j = v^M$ and $g_{sf}(t,\zeta_j,u_{j-1}) = [g_{sf}(t,\zeta_j,u_{j-1})]_{u_m}^0 = g_{us}(\zeta_j,u_{j-1})$, then $\dot\sigma_j = 0$,*

(c) *If $v_j = v_{j-1} \geq 0$ and $g_{sf}(t,\zeta_j,u_{j-1}) = g_{us}(\zeta_j,u_{j-1})$, then $\dot\sigma_j = 0$ and $u_j = u_{j-1}$,*

(d) *If $v_j = v^M$, then $g_{us}(\zeta_j,u_{j-1}) \geq [g_{us}(\zeta_j,u_{j-1})]_{u_m}^0 = 0$ only if*

$$v_{j-1} \geq \underline{v} \triangleq \frac{-u_m v^M}{-u_m + \sigma_0 u_M}.$$

Proof. For the sake of conciseness, we drop the arguments of the functions wherever it causes no confusion.

(a) For $v_j = 0$, the claim readily follows from the definition of g_{us}. For fixed $\sigma_j \geq 1$, $v_j \geq v_{j-1} \geq 0$ and $v_j > 0$, we see that g_{us} is maximized and minimized when $u_{j-1} = u_M$ and $u_{j-1} = u_m$, respectively. The result then follows by observing, after some computations, that $g_{us}(\zeta_j,u_M) - u_M \leq 0$ and $g_{us}(\zeta_j,u_m) - u_m \geq 0$.

(b) and (c) From (7) observe that

$$\begin{aligned}\dot\sigma_j &= \frac{v_{j-1} - v_j - \sigma_j \dot{\mathcal{D}}(v_{j-1}(t), v_j(t))}{\mathcal{D}(v_{j-1}(t), v_j(t))} \\ &= \frac{v_{j-1} - v_j - \frac{\sigma_j}{-u_m}(v_j u_j - v_{j-1}u_{j-1})}{\mathcal{D}(v_{j-1}(t), v_j(t))}\end{aligned}$$

where we have used the fact that $v_j \geq v_{j-1}$ in the coupling set $\mathcal{C}_s$. Claim (b) now follows by substituting $u_j = g_{sf} = g_{us}$ and using the definition of g_{us}. A similar argument can be used to show claim (c).

(d) Setting $v_j = v^M$ in the definition of g_{us} and using the fact that $g_{us} \geq 0$, we have

$$v_{j-1} \geq \frac{-u_m v^M}{-u_m + \sigma_j u_{j-1}}.$$

To obtain the necessary condition on v_{j-1}, we set $u_{j-1} = u_M$ and $\sigma_j = \sigma_0$, the maximum values for each. □

Lemma IV.1 identifies conditions under which we can describe the behavior of the vehicles when they are in the coupling set. The unsaturated controller has been designed so as to ensure that the safety ratio is maintained at a constant level. We know from claim (a) that g_{us} respects the control constraints. Thus, in claim (b), we see that when $v_j < v^M$ and $g_{sf} = g_{uc}$, the control action would not violate the velocity constraints and hence ensures that the safety ratio remains constant, i.e., $\dot\sigma_j = 0$. Similarly, when $v_j = v^M$, if the control action $g_{sf} = g_{uc}$ and it happens to be non-positive, then again the velocity constraints would not be violated and we have $\dot\sigma_j = 0$. Claim (c) is a special case of (b) with $v_j = v_{j-1}$. In this special case, we further have that the relative acceleration, and hence also the relative velocity, are zero. Finally, claim (d) is a necessary condition on the velocity of vehicle $j-1$ for g_{us} and the saturated $[g_{us}]_{u_m}^0$ to differ when $v_j = v^M$. In other words, if $v_{j-1} \leq \underline{v}$, then from claims (d) and (b) we have that $\dot\sigma_j = 0$. It is worth noting that, while considered separately the conditions in each case of Lemma IV.1 might appear restrictive, when considered all together they paint a fairly general picture.

As we have seen in Lemma IV.1 and its interpretation, the unsaturated controller g_{us} has been designed with the aim of ensuring that the safety ratio remains constant. Thus, it would be interesting to determine conditions under which the use of g_{us} guarantees string stability. The next result states that, in fact, this is the case in the absence of velocity constraints and assuming that the leading vehicle's velocity is uniformly upper bounded in time.

Proposition IV.2. *(Unsaturated controller g_{us} ensures string stability in the absence of velocity constraints). Consider two vehicles $j-1$ and j, with vehicle j following $j-1$. Suppose that the velocity of the leading vehicle $j-1$ is uniformly lower and upper bounded in time by 0 and a constant $\bar{\mathcal{V}}$, respectively. Further suppose that no bound on the velocity of vehicle j is imposed. If the initial condition are such that $v_j(0) \geq v_{j-1}(0) \geq 0$ and $\sigma_j(0) \in [1, \sigma_0]$, then the control policy $u_j = g_{us}$ ensures that:*

(a) *safety is guaranteed for all time t by ensuring $\sigma_j(t) = \sigma_j(0) \geq 1$,*
(b) *$v_j \leq \sqrt{(v_j(0))^2 - (v_{j-1}(0))^2 + \bar{\mathcal{V}}^2}$, for all $t \geq 0$,*
(c) *v_j asymptotically approaches v_{j-1},*
(d) *$x_{j-1} - x_j$ asymptotically converges to $\sigma_j(0)L \leq \sigma_0 L$.*

Proof. If there exists s such that $v_j(s) = v_{j-1}(s) = 0$, then the result is trivially true because, by definition, $u_j(t) = u_{j-1}(t)$ for all $t \geq s$. Thus, in the remainder of the proof we assume that $v_j(t) > 0$ for all t in addition to the fact that $v_j(t) \geq v_{j-1}(t) \geq 0$. Lemma IV.1(b) directly guarantees claim (a).

Now, let $e_j \triangleq v_j - v_{j-1}$ and observe that

$$\begin{aligned} \dot{e}_j &= u_j - u_{j-1} = g_{us} - u_{j-1} \\ &= -\left(\frac{-u_m}{\sigma_j}\right)\left(\frac{e_j}{v_j}\right)\left(\frac{\sigma_j u_{j-1} - u_m}{-u_m}\right) \\ &= -\left(\frac{\sigma_j(0) u_{j-1} - u_m}{\sigma_j(0) v_j}\right) e_j, \end{aligned} \tag{11}$$

where in the last step we used $\sigma_j = \sigma_j(0) \geq 1$. Based on the discussion above, we exclude the case of $v_j = 0$. Thus, under such conditions, $e_j = 0$ is invariant. This implies that $e_j \geq 0$ for all $t \geq 0$ given the assumption on the initial condition. Also note that $u_m < 0$, $u_{j-1} \geq u_m$ and $v_j \geq 0$. As a result the following observations hold:

- $\dot{e}_j > 0$ only if $u_{j-1} \in [u_m, u_m/\sigma_j(0))$,
- $\dot{v}_j = u_j > 0$ only if $u_{j-1} > 0 > u_m/\sigma_j(0)$,
- $\dot{v}_j = u_j > 0$ only if $\dot{e}_j < 0$.

The first observation follows from (11). The second is obtained by setting $g_{us} > 0$, which implies

$$u_{j-1} > \frac{-u_m e_j}{\sigma_j(0) v_{j-1}} \geq 0.$$

The final observation follows from the first two. Thus, at any given time, we have at least one of e_j or v_j non-increasing. Motivated by this, we next show v_j is bounded. Indeed, from (7) and the fact $\sigma_j(t) = \sigma_j(0)$,

$$\sigma_j(0)\dot{\mathcal{D}}(v_{j-1}(t), v_j(t)) = -e_j \leq 0,$$

which implies that $\mathcal{D}$ is non-increasing because $e_j \geq 0$. Then, from (4) and the fact $v_j(t) \geq v_{j-1}(t)$ for all t,

$$(v_j(t))^2 \leq (v_j(0))^2 - (v_{j-1}(0))^2 + (v_{j-1}(t))^2,$$

from which claim (b) follows.

Since $v_{j-1} \leq v_j$ for all $t \geq 0$, the inter-vehicular distance $(x_{j-1} - x_j)$ is monotonically non-increasing. Further, since $\sigma_j = \sigma_j(0)$, we see from (7) and (4) that $(x_{j-1} - x_j)$ is uniformly lower bounded by $\sigma_j(0)L$. Thus,

$$x_{j-1}(t) - x_j(t) = -\int_0^t e_j(s)\mathrm{d}s + x_{j-1}(0) - x_j(0)$$

must asymptotically converge to a finite constant. Now, notice that $|\dot{e}_j|$ is uniformly upper bounded due to the bounds on u_{j-1} and u_j. Hence, e_j is uniformly continuous. Then, claim (c) follows from Barbalat's Lemma, cf. [19].

Finally, we know that $\sigma_j(t) = \sigma_j(0) \leq \sigma_0$ for all t. Further, as v_j approaches v_{j-1}, the safe-following distance $\mathcal{D}(v_{j-1}(t), v_j(t))$ approaches L (cf. (4)). Then, claim (d) follows from the definition of the safety ratio (7). □

As a consequence of Proposition IV.2, any string with a finite number of vehicles can be stabilized using the controller g_{us} if each pair of consecutive vehicles is initially in the coupling set and if the velocity of the first vehicle in the string is uniformly upper bounded in time.

The next result states that if at any time instant the optimal controller does not exist (because the vehicle has been slowed down by preceding vehicles), then a vehicle not in the coupling set moves at the maximum speed.

Lemma IV.3. *(If the uncoupled optimal controller does not exist then the vehicle exits the coupling set at maximum speed). Let t_1 be any time such that $\zeta_j(t_1) \in \mathcal{C}_s$ and $\zeta_j(t) \notin \mathcal{C}_s$ for $t \in (t_1, t_1 + \delta)$ for some $\delta > 0$. If $\nexists \mathcal{F}_j$ at time t_1, then $v_j(t) = v^M$ for all $t \in [t_1, t_1 + \delta)$.*

Proof. Under the hypotheses of the result, and as a consequence of Lemma IV.1(c), the only way $v_j(t_1) = v_{j-1}(t_1)$ is possible is if $g_{sf} = g_{uc} < u^M$ at t_1, i.e., $\exists \mathcal{F}_j$. However, by assumption $\nexists \mathcal{F}_j$ at time t_1, meaning $g_{uc} = u^M$. Thus, it must be that $v_j(t_1) > v_{j-1}(t_1)$. By definition of t_1, we then conclude that $\sigma_j(t_1) = \sigma_0$. Next, at t_1, since $\nexists \mathcal{F}_j$ it means $g_{uc} = u_M$ and thus $g_{sf} = g_{us}$. Then, from Lemma IV.1(b), we see that $v_j(t_1) < v^M$ is not possible and that in fact $v_j(t_1) = v^M$ and $g_{sf} = g_{us} > [g_{sf}]_{u_m}^0 = 0$. During the interval $(t_1, t_1 + \delta)$, we see from the second case of (10) that $u_j = [g_{uc}]_{u_m}^0 = [u_M]_{u_m}^0 = 0$, which proves the result. □

This result is useful in our forthcoming analysis to bound the arrival times of consecutive vehicles to the target region.

B. Guarantees on vehicle approach times to target region

In this section we provide guarantees on the vehicle approach times to the target region under the `local vehicular` controller. Our main result states that the prescribed approach time of a vehicle can be met provided it is sufficiently far from the actual approach time of the previous vehicle in the string. If this is not the case, then the result provides an upper bound on the difference between the actual approach times.

To precisely quantify the upper bound, we introduce below the quantity T^{iat}. To justify its definition, we first need to introduce some useful concepts. Let $\mathcal{D}^{\text{nom}} \triangleq \mathcal{D}(\nu^{\text{nom}}, v^M)$, which has the interpretation of a safe inter-vehicle distance given a vehicle is traveling at the maximum allowed speed v^M and the vehicle preceding it is traveling at a speed greater than or equal to ν^{nom}. Recall that we require that each vehicle maintain a velocity of at least ν^{nom} as it approaches the target region and subsequent to it. Given the monotonicity properties of the safe-following distance function $\mathcal{D}$ defined in (4), we see that $\mathcal{D}^{\text{nom}}$ is an upper bound on the safe-following distance for any pair of consecutive vehicles $j-1$ and j for all time subsequent to the approach time of vehicle $j-1$, i.e., for all $t \geq T^a_{j-1}$. Thus, if the following vehicle j is within the coupling set with vehicle $j-1$ at the time of its approach, T^a_j, then we show in the proof of the next result that the inter-approach time $T^a_j - T^a_{j-1}$ is upper bounded by $\sigma_0 T^{\text{nom}}$, where

$$T^{\text{nom}} \triangleq \mathcal{D}^{\text{nom}}/\nu^{\text{nom}}, \tag{12}$$

which we call the *nominal safe inter-vehicle approach time*.

If, instead, vehicles $j-1$ and j do not belong to the coupling set at T^a_j, and $T^a_j > \tau_j$, then from Lemma IV.3 we know that if t_e is the moment when $\nexists \mathcal{F}_j$ and vehicle j exits (never to enter again) the coupling set with vehicle $j-1$, then $v_j = v^M$ for all $t \in [t_e, T^a_j]$. Note also that, by definition, $\sigma_j(t_e) = \sigma_0$. Thus, letting $x_{j-1}(t_e) = -d$ and $v_{j-1}(t_e) = v$, we see from (7) that $x_j(t_e) = -(d + \sigma_0 \mathcal{D}(v, v^M))$. Hence,

$$T^a_j = \frac{d + \sigma_0 \mathcal{D}(v, v^M)}{v^M},$$

and as a result $T^a_j - T^a_{j-1} \leq \mathcal{L}(d, v)$, with

$$\mathcal{L}(d, v) \triangleq \frac{d + \sigma_0 \mathcal{D}(v, v^M)}{v^M} - \mathcal{T}(d, v), \tag{13}$$

where $\mathcal{T}$ is as defined in (5) and gives the earliest possible approach time given the distance to go and the current velocity. With this discussion in place, we are ready to define

$$T^{iat} \triangleq \max\{\sigma_0 T^{\text{nom}}, \max_{\substack{d \geq \frac{(\nu^{\text{nom}})^2 - v^2}{2u_M}, \\ v \in [\underline{v}, \nu^{\text{nom}}]}} \mathcal{L}(d, v)\}, \tag{14}$$

where $\underline{v}$ is given in Lemma IV.1(d). This time also plays a key role in uniformly upper bounding (independently of the initial conditions) the difference between the actual approach times of consecutive vehicles if they are not in the coupling set when reaching the target region. The constraints on d and v in (14) essentially constitute, as shown in the proof of the next result, a sufficient condition for the occurrence of the case in which the vehicles $j-1$ and j do not belong to the coupling set at T^a_j, and $T^a_j > \tau_j$.

We are now ready to state formally the first result of this section.

Proposition IV.4. *(Inter-approach times of vehicles at the target region). For any vehicle $j \in \{2, \ldots, N\}$, suppose that (6) holds, $\tau_j \in [\tau^e_j, \tau^l_j]$, and $v_{j-1}(T^a_{j-1}) \geq \nu^{nom}$. Then, $v_j(T^a_j) \geq \nu^{nom}$ and*

(a) *if $\tau_j - T^a_{j-1} \leq T^{iat}$, then $T^a_j - T^a_{j-1} \leq T^{iat}$,*
(b) *if $\tau_j - T^a_{j-1} \geq T^{iat}$, then $T^a_j = \tau_j$.*

Proof. First note that initially at $t = 0$, Lemma III.4 guarantees that $\exists \mathcal{F}_j$. Next, notice from the definition of the controller (10) that $u_j(t) \leq g_{uc}$ for all $t \geq 0$. Further notice that if at some time t_1, $\nexists \mathcal{F}_j$, then it remains $\nexists \mathcal{F}_j$ for all $t \geq t_1$ for otherwise it means there exists some control policy starting from $t = t_1$ such that $T^a_j = \tau_j$ and $v_j(T^a_j) \geq \nu^{\text{nom}}$ and Remark III.6 guarantees $\exists \mathcal{F}_j$ at $t = t_1$. From this discussion, we deduce that $T^a_j \geq \tau_j$ for each vehicle j.

(a) There are two cases - either the uncoupled optimal controller exists until the vehicle reaches the target region or it becomes infeasible earlier. We consider each of these cases separately. In the first case, notice that for any vehicle $j \in \{2, \ldots, N\}$, if $\exists \mathcal{F}_j$ at $t = T^a_j$, then it follows from the definition of T^a_j that $T^a_j = \tau_j$ and $v_j(T^a_j) \geq \nu^{\text{nom}}$, which means claim (a) is true in the first case.

Next, we consider the case when $\nexists \mathcal{F}_j$ first occurs at some time $t_f < T^a_j$. Clearly, $\zeta_j(t_f) \in \mathcal{C}_s$. Now, there are two sub-cases - either $\zeta_j(T^a_j) \in \mathcal{C}_s$ or $\zeta_j(T^a_j) \notin \mathcal{C}_s$. In the first sub-case, we have by definition that $\sigma_j(T^a_j) \leq \sigma_0$ and $v_j(T^a_j) \geq v_{j-1}(T^a_j)$. Then, the fact that $v_{j-1}(t) \geq \nu^{\text{nom}}$ for all $t \geq T^a_{j-1}$ implies

$$\begin{aligned} x_{j-1}(T^a_j) - x_j(T^a_j) &= \sigma_j(T^a_j) \cdot \mathcal{D}(v_{j-1}(T^a_j), v_j(T^a_j)) \\ &\leq \sigma_0 \cdot \mathcal{D}^{\text{nom}}, \end{aligned}$$

where we have used the definition of $\mathcal{D}^{\text{nom}}$ and the monotonicity properties of the safe-following distance function $\mathcal{D}$ in deriving the inequality. Now, imagine a virtual particle rigidly fixed to vehicle $j-1$ at a distance of $\sigma_0 \mathcal{D}^{\text{nom}}$ behind it. Since $v_{j-1}(t) \geq \nu^{\text{nom}}$ for all $t \geq T^a_{j-1}$, we can then conclude that $T^a_j - T^a_{j-1} \leq \sigma_0 \frac{\mathcal{D}^{\text{nom}}}{\nu^{\text{nom}}} = \sigma_0 T^{\text{nom}} \leq T^{iat}$.

We are then left with the sub-case when $\zeta_j(T^a_j) \notin \mathcal{C}_s$. Thus, suppose that there exists $t_e \geq t_f$ such that $\zeta_j(t) \notin \mathcal{C}_s$ for all $t \in (t_e, T^a_j]$ and $\zeta_j(t_e) \in \mathcal{C}_s$. From Lemma IV.3, it follows that $v_j(t) = v^M$ for all $t \in [t_e, T^a_j]$. Thus, as we have seen in (13), $T^a_j - T^a_{j-1} \leq \mathcal{L}(d, v)$ with $x_{j-1}(t_e) = -d$ and $v_{j-1}(t_e) = v$. Thus, now it remains to justify the constraints on d and v in (14). Given the assumption that $v_{j-1}(T^a_{j-1}) \geq \nu^{\text{nom}}$ it follows that $d \geq \frac{(\nu^{\text{nom}})^2 - v^2}{2u_M}$, which is the minimum distance traversed as the velocity of a vehicle evolves from v to ν^{nom}. Next, by the definition of t_e, note that $\sigma_j(t_e) = \sigma_0$ and $\sigma_j(t) > \sigma_0$ for all $t \in (t_e, T^a_j]$, implying that $\dot{\sigma}_j(t_e) > 0$. From Lemma IV.1(b)-(d), we then deduce that $v_{j-1}(t_e) \geq \underline{v}$. Finally, notice that

$$\begin{aligned} x_{j-1}(T^a_j) - x_j(T^a_j) &\leq x_{j-1}(t_e) - x_j(t_e) \\ &= \sigma_0 \mathcal{D}(v_{j-1}(t_e), v_j(t_e)), \end{aligned}$$

where the inequality follows from $v_{j-1}(t) \leq v_j(t) = v^M$ for all $t \in [t_e, T^a_j]$. Consequently, if $v_{j-1}(t_e) = v \geq \nu^{\text{nom}}$, then $\mathcal{D}(v, v^M) \leq \mathcal{D}(\nu^{\text{nom}}, v^M)$ and hence we deduce $T^a_j - T^a_{j-1} \leq \sigma_0 T^{\text{nom}}$, which justifies the final constraint in (14) and hence proves claim (a).

(b) The main argument for the proof of this claim is that the uncoupled optimal controller exists until the vehicle reaches the target region, which we show by contradiction. Suppose that $\nexists F_j$ at T^a_j. Then as in the proof of claim (a), we see that $T^a_j - T^a_{j-1} \leq T^{iat}$. However, Lemma III.4 guarantees that if $T^a_j = \tau_a$ is feasible then so is $T^a_j = \tau_b$ for any $\tau_b \geq \tau_a$. Using this for the case $\tau_a = T^a_j$ and $\tau_b = \tau_j$, we would deduce that $T^a_j = \tau_j$ is feasible, which is a contradiction. The rest of the proof is the same as in the first case of the proof of (a). □

Note that in (14), T^{iat} is defined as the solution of a maximization problem. However, since the maximization problem involves only the parameters of the system, it could be solved offline. In fact, we can give an analytical expression for T^{iat}, which we present in the next result.

Corollary IV.5. *(Analytical expression for T^{iat}).*

$$T^{iat} = \begin{cases} \sigma_0 T^{nom}, & \text{if } \underline{v} > \nu^{nom}, \\ \max\{\sigma_0 T^{nom}, T^{fol}(\underline{v})\}, & \text{if } \underline{v} \le \nu^{nom}, \end{cases} \quad (15)$$

where

$$T^{fol}(v) \triangleq \frac{(\nu^{nom})^2 - v^2}{2u_M v^M} + \frac{\sigma_0 \mathcal{D}(v, v^M)}{v^M} + \frac{\nu^{nom} - v}{u_M}.$$

Proof. The case of $\underline{v} > \nu^{\mathrm{nom}}$ follows directly from the fact that maximization $\mathcal{L}$ in (14) is infeasible. Thus, now we assume $\underline{v} \le \nu^{\mathrm{nom}}$. By direct computation, we see that

$$\frac{\partial \mathcal{L}(d,v)}{\partial d} \begin{cases} < 0, & 2u_M d < (v^M)^2 - v^2 \\ = 0, & 2u_M d \ge (v^M)^2 - v^2. \end{cases}$$

Thus, it follows that

$$\begin{aligned} \max_{\substack{d \ge \frac{(\nu^{\mathrm{nom}})^2 - v^2}{2u_M}, \\ v \in [\underline{v}, \nu^{\mathrm{nom}}]}} \mathcal{L}(d,v) &= \max_{\substack{d = \frac{(\nu^{\mathrm{nom}})^2 - v^2}{2u_M}, \\ v \in [\underline{v}, \nu^{\mathrm{nom}}]}} \mathcal{L}(d,v) = \max_{v \in [\underline{v}, \nu^{\mathrm{nom}}]} T^{\mathrm{fol}}(v) \\ &= T^{\mathrm{fol}}(\underline{v}), \end{aligned}$$

where the final equality follows from the fact that T^{fol} is a decreasing function of v. ☐

The next result summarizes the guarantees provided by the `local vehicular` controller (10) regarding the satisfaction of the constraints on safety and approach times.

Theorem IV.6. *(Provably safe sub-optimal distributed control under finite-time constraints). Consider a string of vehicles $\{1, \ldots, N\}$ whose dynamics are described by (1) under the `local vehicular` controller (10). Assume that $x_1(0) \le (v^M)^2/2u_m - (\nu^{nom})^2/2u_M$, that $\tau_j \in [\tau_j^e, \infty)$ and that the vehicles are in a safe configuration initially, ($\sigma_j(0) \ge 1$ for all $j \in \{2, \ldots, N\}$). Then,*

- (a) *inter-vehicle safety is ensured for all vehicles and for all time subsequent to 0 (i.e., $\sigma_j(t) \ge 1$ for all $j \in \{2, \ldots, N\}$ and $t \ge 0$),*
- (b) *the first vehicle approaches the target region at τ_1, each vehicle travels with a velocity of at least ν^{nom} at the time of approaching the target region and subsequent to it and*
- (c) *for each $j \in \{2, \ldots, N\}$, if $\tau_j - T_{j-1}^a \le T^{iat}$ then $T_j^a - T_{j-1}^a \le T^{iat}$. Alternatively, if $\tau_j - T_{j-1}^a \ge T^{iat}$, then $T_j^a = \tau_j$.*

Proof. (a) Note that for $\sigma_j \in [1, \sigma_0]$, if $\zeta_j \in \mathcal{C}_s$, then σ_j either stays constant, in the case of Lemma IV.1(c), or increases, in the case of Lemma IV.1(d). If on the other hand $\zeta_j \notin \mathcal{C}_s$, then it means $v_j < v_{j-1}$ and $x_{j-1} - x_j$ increases while $\mathcal{D}(v_{j-1}, v_j)$ stays constant at L and thus σ_j increases. Thus $\sigma_j(t) \ge 1$ is guaranteed for all vehicles $j \in \{2, \ldots, N\}$ and for all $t \ge 0$.

(b) Since there is no vehicle in front of vehicle 1, $u_1 = g_{uc}$ for all t. Initial feasibility then guarantees that $T_1^a = \tau_1$ and $v_1(T_1^a) \ge \nu^{\mathrm{nom}}$.

Claim (c) follows directly from Proposition IV.4 and by using induction. ☐

Note that we have not guaranteed optimality of our proposed solution and in general it is only suboptimal. However, the uncoupled optimal control mode ensures that the overall distributed controller is optimum seeking for each individual vehicle.

C. Integration with intelligent intersection management

Here we elaborate on the application to intelligent intersection traffic management, cf. Remark II.2, of our distributed control design for a string of vehicles under finite time constraints. We envision a system where each vehicle or groups of vehicles communicate their aggregate information to a central intersection manager. The intersection manager seeks to optimize the schedule of the usage of the intersection by the vehicles. With the information received, the manager schedules an intersection occupancy time interval to each group of vehicles. The vehicles belonging to each group then apply the `local vehicular` controller (10) in order to satisfy the prescribed schedule while also maintaining safety. The aggregate information required by the central intersection manager from each group of vehicles has two pieces: constraints on the approach time τ_1 of the first vehicle in the group and a bound on the occupancy time $\bar{\tau}^{\mathrm{occ}} \ge T_N^{\mathrm{exit}} - \tau_1$ of the intersection that could be guaranteed by the `local vehicular` controller (10). We discuss next how to compute each element.

1) Constraints on approach time of the first vehicle: The constraints on τ_1 could be computed by ignoring other vehicles in the group, as in Section III-B. However, in doing so, ignoring the initial conditions of the other vehicles in the group poses the risk of lengthening the guaranteed upper bound $\bar{\tau}^{\mathrm{occ}}$ on the occupancy time. The reasoning for this is better explained in terms of earliest times of approach at the intersection of the vehicles. If τ_j^e for some $j > 1$ is significantly greater than τ_1^e, then having the vehicle 1 slow down to approach the intersection at a time later than τ_1^e will allow the string of vehicles to meet a smaller guaranteed upper bound $\bar{\tau}_i^{\mathrm{occ}}$ on the occupancy time.

Given this observation, we propose the following alternative way of computing the constraints on the approach time of the first vehicle. Recalling the interpretation of T^{nom} as the nominal inter-vehicle approach time of vehicles in the group, we see that the earliest time of approach for vehicle j puts a constraint on the earliest time of approach of the group, i.e., vehicle 1, to be no less than $\tau_j^e - (j-1)\mathcal{A}T^{\mathrm{nom}}$, where $\mathcal{A} \in [0,1]$ is a design parameter that determines the aggressiveness with which the prescribed approach times for the vehicles are spaced. The smaller the value of $\mathcal{A}$, smaller is the gap between the prescribed approach times and hence greater is the aggressiveness with which the vehicles are forced to enter the coupling set and use safe-following control. Hence, we define the *earliest time T_1^e of approach* for the group of vehicles as

$$T_1^e \triangleq \max\{\tau_j^e - (j-1)\mathcal{A}T^{\mathrm{nom}} : j \in \{1, \ldots, N\}\}. \quad (16)$$

Similarly, we can also compute the *latest time of approach* T_1^l for the group of vehicles. Note that, under the assumptions of Lemma III.4, $T_1^l = \infty$. Further, for each vehicle $j \in \{1, \ldots, N\}$ in the group, we have the intersection manager prescribe

$$\tau_j \triangleq \tau_1 + (j-1)\mathcal{A}T^{\text{nom}}, \tag{17}$$

so that the only variable it must compute is τ_1. Thus, we see that the parameter $\mathcal{A}$ influences the aggressiveness with which the vehicles are driven into the safe-following mode. For example, $\mathcal{A} = 0$ means that $\tau_j = \tau_1$ for all j, which necessarily means that the each vehicle must enter the safe-following mode at least once. For higher values of $\mathcal{A}$, there is a greater chance of the uncoupled controller g_{uc} for vehicle j being feasible until its approach time T_j^a. Given the constraints that the scheduler takes into account, we have $\tau_1 \in [T_1^e, T_1^l]$. This, together with (16), implies that $\tau_j \in [\tau_j^e, \tau_j^l]$, i.e., the sequence $\{\tau_j\}_{j=1}^N$ of approach times prescribed by the intersection manager is feasible ignoring the safety constraints.

2) Guaranteed bound on occupancy time of intersection: Given the sequence of approach times prescribed in (17) by the intersection manager, the following result builds on Proposition IV.4 to provide a guaranteed upper bound on the occupancy time of the target region by the group of vehicles.

Corollary IV.7. *(Guaranteed upper bound on occupancy time of the group of vehicles). For the string of vehicles $\{1, \ldots, N\}$, suppose $\tau_1 \geq \tau_1^e$, where τ_1^e is given by (16), and τ_j for $j \in \{2, \ldots, N\}$ satisfies (17). Then, the occupancy time $\tau^{occ} \triangleq T_N^{\text{exit}} - T_1^a$ is upper bounded as $\tau^{occ} \leq \bar{\tau}^{occ}$, where*

$$\bar{\tau}^{occ} = (N-1)T^{iat} + \max\left\{\frac{L+\Delta}{\nu^{nom}}, T^{iat}\right\}. \tag{18}$$

Proof. From Theorem IV.6, we know that $T_1^a = \tau_1$. We also know that $T_j^a \geq \tau_j$ for each $j \in \{2, \ldots, N\}$. Thus, as a result of (17), we know that $\tau_j - T_{j-1}^a \leq T^{\text{nom}} < T^{iat}$ for all $j \in \{2, \ldots, N\}$. Hence, from Proposition IV.4, we see that the last vehicle N approaches the target region at time T_N^a satisfying $T_N^a \leq T_1^a + (N-1)T^{iat}$. Since each vehicle travels with a velocity of at least ν^{nom} after approaching the intersection, the vehicle N (and thus the group of vehicles) exits the intersection no later than $T_N^a + \frac{L+\Delta}{\nu^{\text{nom}}}$. That is,

$$T_N^{\text{exit}} \leq T_N^a + \frac{L+\Delta}{\nu^{\text{nom}}} \leq T_1^a + (N-1)T^{iat} + \frac{L+\Delta}{\nu^{\text{nom}}},$$

from which the result follows. □

The reasoning for the inclusion of T^{iat} in the second term of (18) is as follows. There may be a second group of vehicles that uses the intersection immediately after the first group. Thus, we would like to have a safe-following distance between the last vehicle of the first group and the first vehicle of the second group even as it approaches the intersection at its assigned time. The inclusion of the term T^{iat} ensures that if $\tau_{N+1} \geq \tau_1 + \bar{\tau}^{\text{occ}}$ then $\tau_{N+1} - T_N^a \geq T^{iat}$, where $N+1$ is the index of the first car in the second group of vehicles. Then, from Proposition IV.4(b), it follows that $T_{N+1}^a = \tau_{N+1}$. This helps in scheduling the intersection usage by several groups of vehicles with just the aggregate data of τ_1 and $\bar{\tau}^{\text{occ}}$ for each group.

V. Simulations

This section presents simulations of the vehicle string evolution under the proposed `local vehicular` controller. Table I specifies the system parameters employed in the simulations (T^{nom} and T^{iat} are computed according to (12) and (14), while the remaining parameters are design choices or are typical of cars and arterial roads). All the units are given in SI units. For better intuition, v^M and ν^{nom} are equivalently 60km/h and 48km/h, respectively. We present five sets of simulations,

TABLE I
System parameters

Parameter	Symbol	Value
Car length	L	4m
Target region length	Δ	12m
Max. speed limit	v^M	16.667m/s
Max. accel.	u_M	3m/s^2
Min. accel.	u_m	-4m/s^2
Nominal speed of crossing	ν^{nom}	13.333m/s
Parameter in (9)	σ_0	1.2
Nominal inter-vehicle approach time	T^{nom}	≈ 1.24s
Upper bound on inter-vehicle approach time	T^{iat}	≈ 1.59s

labeled Sim1 to Sim5. In all of them, the number of vehicles is $N = 8$. The initial conditions are randomly generated so that initial safety, $\sigma_j(0) \geq 1$, is satisfied for all $j \in \{2, \ldots, N\}$ and (6) holds for all $j \in \{1, \ldots, N\}$. In all simulations but Sim5, the initial conditions are the same, with the only distinguishing factor being how the prescribed approach times τ_j are determined.

In Sim1, shown in Figure 3, the prescribed approach times are randomly generated, with the only constraints being $\tau_j \geq \tau_j^e$. The choice of random τ_j does not in general result in

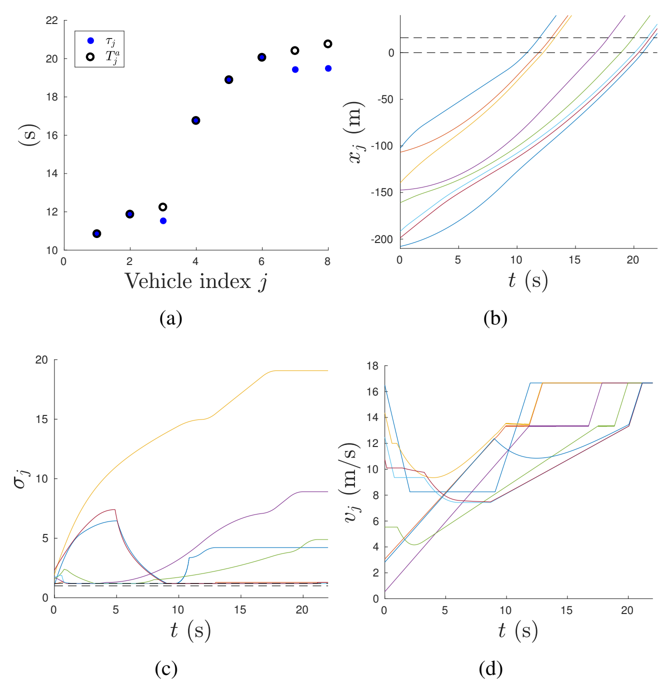


Fig. 3. Results for Sim1. (a) Prescribed and actual approach times of the vehicles in the string. (b) Evolution of the position of the vehicles. The region between the dotted lines is the target region. (c) Evolution of the safety ratios. The dotted lines are at $\sigma = 1$ and $\sigma_0 = 1.2$. (d) Evolution of the velocities.

cohesion of the vehicles as they pass through the target region, as can be clearly seen in Figures 3(b)-3(c). The occupancy time in this case is $\tau^{\text{occ}} = 10.88$s. In this case, we do not have either an analytical expression for the bound on the occupancy time (which is why the prescribed approach times must instead be constrained, for example as in (17), in the context of intersection management).

In Sim2 to Sim4, $\tau_1 = T_1^e$ is chosen with T_1^e as in (16) and the remaining τ_j are determined according to (17). In Sim2, shown in Figure 4, we choose $\mathcal{A} = 1$. Figure 4(a) shows that

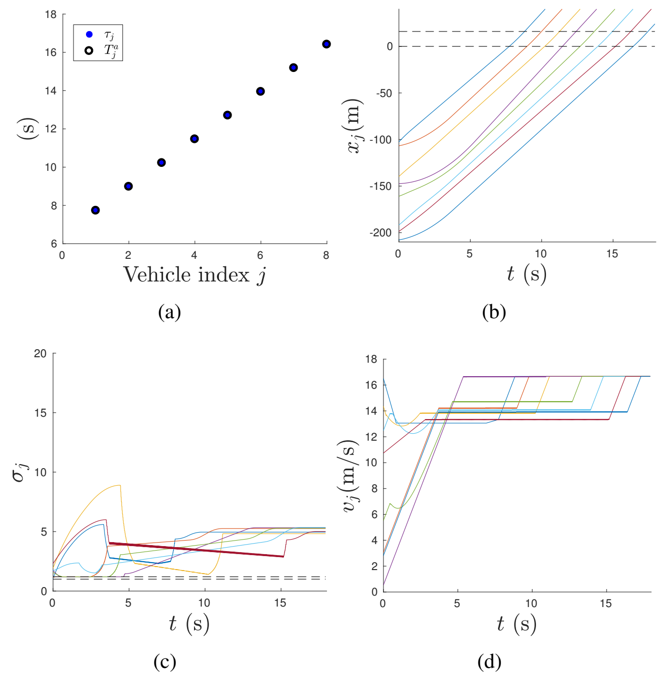


Fig. 4. Results for Sim2. (a) Prescribed and actual approach times of the vehicles in the string. (b) Evolution of the position of the vehicles. The region between the dotted lines is the target region. (c) Evolution of the safety ratios. The dotted lines are at $\sigma = 1$ and $\sigma_0 = 1.2$. (d) Evolution of the velocities.

the prescribed and the actual approach times coincide for each vehicle. In fact, for σ_0 smaller than around 4, we have consistently observed that $T_j^a = \tau_j$ for each vehicle j. Figures 4(b)-4(c) demonstrate the moderate cohesion that is achieved as vehicles cross the target region. The occupancy time and the theoretical upper bound are $\tau^{\text{occ}} = 9.72$s and $\bar{\tau}^{\text{occ}} = 12.72$s, respectively. While each vehicle approaching the target region at its prescribed time is desirable, the occupancy time τ^{occ} is large.

In Sim3, shown in Figure 5, we demonstrate the utility of the tuning parameter $\mathcal{A}$ in (16)-(17). We choose $\mathcal{A} = 0$, resulting in the prescribed approach times being all the same, $\tau_j = \tau_1$ for all j. This necessarily forces vehicles to interact through the coupling set and the safe-following controller aggressively. While the actual approach times are no longer equal to their prescribed values (cf. Figure 5(a)), this specification results in high cohesion of vehicles as they cross the target region (cf. Figures 5(b)-5(c)). An important feature we have observed consistently in simulations with $\mathcal{A}$ smaller than 1 is the synchronization of the velocities (cf. Figure 5(d)). By contrast, the velocity synchronization in Figures 3(d) and 4(d) is an

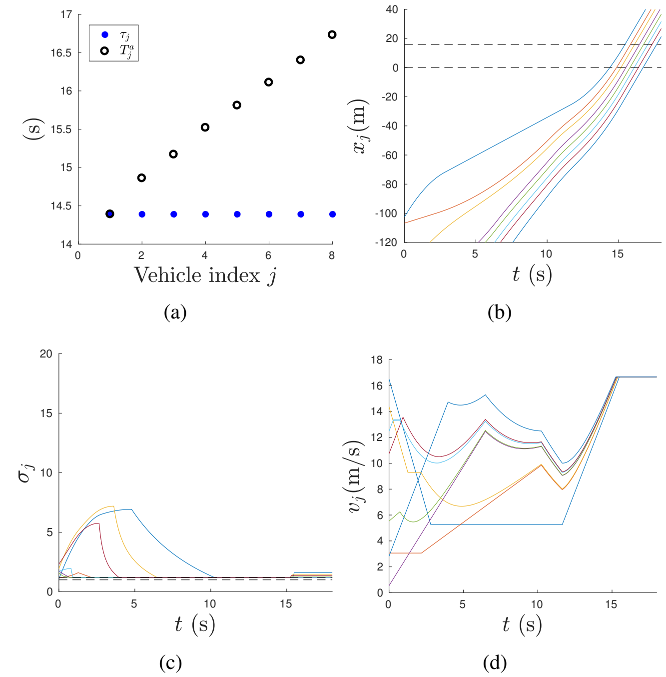


Fig. 5. Results for Sim3. (a) Prescribed and actual approach times of the vehicles in the string. (b) Evolution of the position of the vehicles. The region between the dotted lines is the target region. (c) Evolution of the safety ratios. The dotted lines are at $\sigma = 1$ and $\sigma_0 = 1.2$. (d) Evolution of the velocities.

artifact of the velocity saturation. We do not have analytical proof of this phenomenon however. In Sim3, the occupancy time is much smaller at $\tau^{\text{occ}} = 3.3$s, although our bound on it still remains at $\bar{\tau}^{\text{occ}} = 12.72$s. Note also that the price of a smaller $\mathcal{A}$ is a larger T_1^e, the earliest time that vehicle 1 can approach the intersection. This suggests an interesting trade-off between intersection occupancy time and earliest time of arrival. Figure 6 shows the control profile and the evolution of the control mode of vehicle 8 in Sim3. As expected, the evolution of the control trajectory takes a complex form.

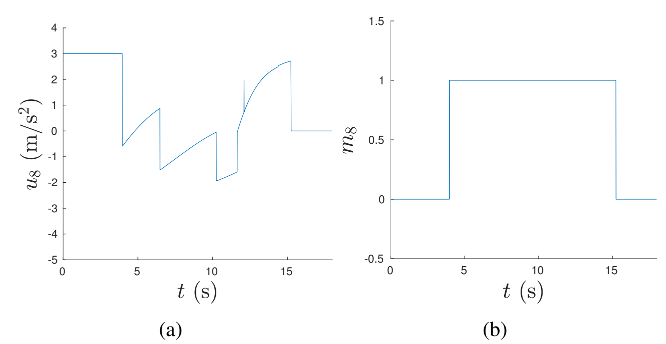
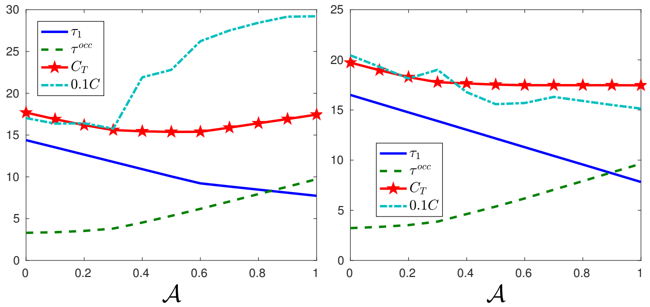


Fig. 6. Results for Sim3. (a) Control profile for vehicle 8. (b) Evolution of the control mode of vehicle 8. $m_8 = 0$ and $m_8 = 1$ indicates vehicle 8 is in the uncoupled mode and safe-following mode respectively.

Finally, we illustrate in Figure 7 the dependence of the prescribed approach time of the first vehicle, the occupancy time, and the time and fuel costs on the tuning parameter $\mathcal{A}$. In Sim4, the initial conditions are the same as in Sims 1-3, while in Sim5 the initial conditions are different. The general trends for τ_1, τ^{occ} and the time cost $C_T = \tau_1 + \tau^{\text{occ}}$ are independent of



(a) (b)

Fig. 7. Results of Sim4 and Sim5. Each plot shows the dependence on the tuning parameter $\mathcal{A}$ of the prescribed approach time of vehicle 1, τ_1, the occupancy time τ^{occ}, the time cost $C_T = \tau_1 + \tau^{\text{occ}}$, and the fuel cost C.

the initial conditions. However, we found the trend of the fuel cost C to be dependent on the initial conditions, as illustrated in Figure 7. Qualitatively, the earliest approach time for the group of vehicles T_1^e, which is also set as τ_1 in the simulations, is decreasing with increasing $\mathcal{A}$, while the occupancy time τ^{occ} is increasing. An explanation for the high dependence of the fuel cost C on the initial conditions is that these are the main factor determining the fraction of time that vehicles spend in the coupled or uncoupled mode. Thus, the value of the parameter $\mathcal{A}$ giving the best performance in terms of C depends on the vehicles' initial conditions.

VI. Conclusions

We have studied the problem of optimally controlling a vehicular string with safety requirements and finite-time specifications on the approach time to a target region. The main motivation for this problem is intelligent management at traffic intersections with networked vehicles. We have proposed a distributed control algorithmic solution which is provably safe (ensuring that even if there was a communication failure, the vehicles could come to a complete stop without collisions) and guarantees that the vehicles satisfy the finite-time specifications under speed limits and acceleration saturation. We have also discussed how the proposed distributed algorithm can be integrated into a larger framework for intersection management for computer controlled and networked vehicles. Finally, we have illustrated our results in simulation. Future work will explore the derivation of tighter bounds on the occupancy time of the intersection, optimizing the trade-off between arrival time of the vehicle string at the intersection and occupancy time, obtaining bounds on the overall fuel cost of the string, refining the design of the safe-following controller, characterizing how the design parameters affect the optimality of our design, exploring robustness guarantees against vehicle addition and removal, and incorporating privacy requirements.

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Pavankumar Tallapragada received the B.E. degree in instrumentation engineering from Shri Guru Gobind Singhji Institute of Engineering and Technology, Nanded, India, in 2005, the M.Sc. (Engg.) degree in instrumentation from the Indian Institute of Science, Bangalore, India, in 2007, and the Ph.D. degree in mechanical engineering from the University of Maryland, College Park, MD, USA, in 2013. From 2014 to 2017, he was a Postdoctoral Scholar in the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA, USA. He is currently an Assistant Professor in the Department of Electrical Engineering, Indian Institute of Science. His research interests include event-triggered control, networked control systems, distributed control and networked transportation, and traffic systems.

Jorge Cortés received the Licenciatura degree in mathematics from Universidad de Zaragoza, Zaragoza, Spain, in 1997, and the Ph.D. degree in engineering mathematics from Universidad Carlos III de Madrid, Madrid, Spain, in 2001. He held postdoctoral positions with the University of Twente, Twente, The Netherlands, and the University of Illinois at Urbana-Champaign, Urbana, IL, USA. He was an Assistant Professor with the Department of Applied Mathematics and Statistics, University of California, Santa Cruz, CA, USA, from 2004 to 2007. He is currently a Professor in the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA, USA. He is the author of Geometric, Control and Numerical Aspects of Nonholonomic Systems (Springer-Verlag, 2002) and co-author (together with F. Bullo and S. Martínez) of Distributed Control of Robotic Networks (Princeton University Press, 2009). He has been an IEEE Control Systems Society Distinguished Lecturer (2010-2014) and is an IEEE Fellow. His current research interests include distributed control, complex networks, opportunistic state-triggered control and coordination, distributed decision making in power networks, robotics, and transportation, and distributed optimization, learning, and games.