

Route Reduction for and Viability Analysis of One-Shot First and Last Mode Transportation

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Abstract

In this paper, we consider a one-shot problem of coordinating the first and last mode transportation, i.e., transportation of demand to or from a hub. We consider a macroscopic formulation and assume a fixed time window for the transportation under the knowledge of the demand and supply volume configurations. The framework is applicable for maximization of either operator profit or social welfare, the latter of which is useful for applications such as disaster response. We use K.K.T. analysis to propose an offline method for reducing the problem size. We also present a model for pricing based on the cost and travel time of the best alternate transportation and present necessary conditions for the feeder service to be viable at a node-level. For the first-mode, we present properties of optimal supply configurations for maximizing profits given a total feeder supply and provide a closed form solution of the absolute maximum “profits” for a given demand. We show that the first-mode problem and the supply optimized last-mode problem are equivalent, which guarantees that many properties carry over to the latter problem. We validate the basic analytical results through a collection of simulations, which includes a disaster relief scenario with real-world data. In simulations, route reduction method is seen to be very effective even for the microscopic formulation as it provides near optimal solutions with significantly greater computational efficiency.

Keywords

First and last-mode transportation, route set reduction, logistics for disaster response, pricing

I. INTRODUCTION

With rapid advances in transportation technologies, large-scale first- and last-mode services (namely *feeder services*) can be planned during operation and in response to demand and supply. The common frameworks for studying such problems has been the vehicle routing problem (V.R.P.) and network flow problems. In V.R.P., the vehicles and demand are typically discrete entities and hence the underlying problem is a mixed integer program, which is often solved with heuristics due its complexity [1], [2]. Further, though there are now many variations of V.R.P., some with very rich features, very often the analysis is limited. Compared to these *microscopic* integer problems, *macroscopic* network flow problems model demands, supplies as volumes and routed vehicles as flows. These problems though less realistic are mostly convex, or even linear, in nature and hence are easier to solve and analyze. Further, a macroscopic formulation can also be seen as a relaxation and hence a solution method for the microscopic formulation. Finally, a macroscopic formulation also allows us to characterize the nature of the solutions, to carry out higher level planning, to quickly determine the limits of and to even analyze economic viability of a transportation service.

Thus, with these aspects in mind, we study a macroscopic network flow [3] formulation of a coordinated and demand responsive feeder service in a region modeled as a network. We consider a one-shot first-mode or *feed-in* problem and the last-mode or *feed-out* problem, wherein all the demand has a common destination and origin, respectively. We call it “one-shot” since there is a single hard time window within which the transportation has to occur. Given the demand and supply configuration on a network, the optimization problem determines optimal routing of the vehicles and servicing of the demand. We also consider the combined problem of optimal supply location and routing. The proposed model includes both maximization of monetary profit and maximization of serviced demand or social welfare as special cases, depending on the choice of the parameters. We achieve this through a pricing model based on a notion of perceived cost. We characterize some properties of the solutions of the optimization problem, propose a method for reducing the problem size, carry out viability analysis and provide illustrative simulations results.

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1) *Applications*: There are many diverse applications of the proposed one-shot transportation model and we discuss a few of them here. Many time-critical logistics problems with humanitarian motives can be modeled as one-shot problems. These include disaster management, transportation of perishable goods and logistics for medical supplies during large scale public health emergencies. As an example, the states on the eastern coast of India - West Bengal, Odisha and Andhra Pradesh - face devastating cyclones quite frequently. The scale of the disaster response is massive and often millions of people are evacuated [4]–[6]. Both the pre and post-cyclone relief are of similar magnitudes affecting millions in the region. Another prime example is that of large-scale emergency transportation of liquid medical oxygen by the Indian Railways across the country in response to shortages during the Covid-19 pandemic [7]. Most of these deliveries had to be carried out only once within a hard time-window.

At the same time, there are also commercial applications such as transportation for sports events or concerts with large foot-fall. For example, a single game of football at Old-Trafford, Manchester is attended by 57000 people on average. A one-shot coordination of a fleet of ride-sharing vehicles to and from the venue to serve the fans could be a potential source of income for an operator. Further, the proposed model has potential applications in multi-modal transportation and logistics systems where the main mode follows a schedule and the first and last modes need to be coordinated to achieve a “fully-coordinated” system. For example, consider a multi-modal courier system serving a “medium” demand city that is serviced by a single *freighter* service a day. The parcel vans in these cities can coordinate to pick-up (deliver) parcels from (to) the warehouses to (from) the transportation hub, such as an airport or train station, following a strict time window as per the *freighter* schedule. Other potential applications are seen in many cases like peak-hour single destination para-transit [8], freight transportation [9] and many other multi-modal setups. Thus, the thorough analysis of the model presented in this paper and the resulting solution methods have a lot of potential uses in a diverse set of applications.

2) *Literature Review*: In the context of routing of transportation services, the vehicle routing problem (V.R.P.) [1], [2], [10]–[16] typically assumes a depot from where one or more vehicles are routed via locations where one or more types of entities are picked up at their origin(s) and dropped off at their destination(s). There are also many variations to the basic V.R.P. like capacitated [13], *split delivery* [17], multiple origin and destination [15], multiple time window [12], [15], or simultaneous pick-up and delivery [16] etc. [11] presents a hard time window V.R.P. and aims at solving the problem using branch and cut algorithm to solve the problem under stochastic service times. [14] presents many vehicle routing scenarios where parameters are dynamic. Most approximate solutions to the V.R.P. employ some form of meta-heuristics [18]. Other major exact solution techniques for the V.R.P. range from branch and bound [19], column-generation methods, set-partitioning methods, etc. [20], [21]. A modern trend in the modelling is also towards the *green-V.R.P.* (G.V.R.P.) [21], [22] which focuses on the routing of vehicles with a fixed time-window along with a constraint on route length. Sharing some similarities with V.R.P. are the ride sharing problem [23]–[27] and dial-a-ride problem [28]–[30]. In these problems, the aim is to match vehicles with the passengers while maximizing operator’s profits in some time windows. Recently, there has also been a growing interest in routing problems with awareness of the demand and the fleet. For example, [31], [32] explore the problem of optimally dispatching taxis based on the location of the taxis and customer requests.

While many routing problems deal with discrete vehicles and discrete entities to be transported, macroscopic models that deal with flows of vehicles and volumes of demand and supply are also common. Though less realistic than discrete models, they allow for computationally easier solutions and greater scope for analysis and higher order planning. In the context of demand anticipative mobility, [33]–[37] aim to match demand and supply by routing autonomous vehicle flows and maximize throughput in the network through a steady-state design of the load-balancing and routing flow rates. Much of the literature on fleet routing is in the context of uni-modal or single hop transportation services. Though multi-modal transportation has been extensively studied, dynamic, demand and supply aware first and last mode service is not sufficiently studied [38], [39].

There have been multiple studies on humanitarian aid planning. Evacuation and distribution of relief material in disaster management scenarios form the prime focus of such studies [40]–[42]. Most of these studies focus on determining the location of the ware-houses and safe-houses given the forecasts though [43] proposes a multi-depot location-routing problem with network failure, multiple use of vehicles, and a fixed relief time. [44] analyzes a last-mile delivery problem to gain insights into the underlying route structures and provides observations on routing aspects that are necessary for planning out disaster relief operations. The results are in the context of three scenarios: time minimization, demand maximization and service comparability. [45] proposes a multi-criterion last-mile routing problem, while [46] considers a capacitated routing problem which minimizes the *maximum response time* for relief to reach the affected regions. In [47], a time-windowed V.R.P. with a single depot is proposed for the purpose of humanitarian aid distribution.

3) *Contributions*: The major contributions in this paper are as follows.

- *Generality of the Mathematical Model:* On the modeling front, we present a unified model that is applicable for both disaster response as well as non-humanitarian applications such as transportation for large public events. We consider routing in a fixed hard time window. In comparison, the demand anticipative mobility problem for macroscopic network flow models in [34]–[37] is concerned with designing steady state flows. Similarly, ride sharing and taxi dispatch do not address fixed horizon planning necessary for single events, such as large public events or for disaster relief. Compared to the problems on disaster relief [43]–[47], Our model can also incorporate priorities for demand at different nodes, depending on say the local severity of a disaster.
- *Utility of the Macroscopic analysis:* Routing problems have been solved with several methods, many of which require solving the macroscopic version of the problem. For example, branch and bound based methods follow a 2 step process, wherein the macroscopic problem is solved first and then using the solutions of the macroscopic problem a good feasible integer solution is calculated. Similarly in set-partitioning methods we need to generate routes which should be as close to the optimal routes as possible. However, in any route based method, the route set starts to explode with increasing time window and also with the increasing number of network nodes. Thus, even the macroscopic problem is not scalable with the number of nodes or with the time window. With the macroscopic analysis of this paper, we try to overcome the problem of time-window based route-set explosion. We observe that the route reduction method proposed in this paper is about 3 times faster at the least within our set of simulations. We also observed that given the proposed pricing model we also have saturation of route sets even with increasing time windows. Further the analytical guarantees of optimality ensures that the accuracy is not compromised. This is also potentially useful when generating route sets in set-partitioning methods. Another potential utility of the macroscopic analysis is in analysing the G.V.R.P. models and formulating exact solutions for the same [22]. For example, [21] can be formulated as a special case of the unified model and therefore all macroscopic analytical guarantees presented in this paper can be utilised for solving the G.V.R.P. Thus, without actually solving the problem in its full integer-constrained flavour we can theoretically predict whether the G.V.R.P. is feasible for a location.
- *Pricing Model and Viability Analysis:* We propose a simple pricing model that is general enough to be applicable for monetary profit maximization as well as social welfare maximization. We carry out mathematical analysis on the parameters to come up with necessary conditions for the viability of the feeder service and for routes with multiple trips to a hub node to be chosen in an optimal solution. The analysis is done on a *node-level* and can be useful in identifying sub-regions where even though *service* is feasible, no optimal solutions *serve* the region. Such analysis is useful for higher level planning.
- *Simulations:* We present a collection of simulations on multiple graphs that validate our analytical results. These simulation results include a realistic scenario of pre-positioning of relief material anticipating the devastation caused by a cyclone. We construct the graph and various other parameters in this simulation based on data from Odisha, a state in India that is prone to severe cyclones.

In the preliminary version [48] of this paper, we considered only the feed-in problem and supply location problem. Here, we additionally provide results for the feed-out problem and introduce a model for carrying out the analysis on the viability of the feeder service. Moreover, here we also provide the proofs missing in [48]. We also have an expanded simulation section where we compare the macroscopic and microscopic versions of the problem. Furthermore, we present a realistic application of planning the relief distribution in anticipation of a cyclone.

4) *Organization:* The rest of the paper is organized as follows - we set up the *one-shot* problem in Section II. In Section III we describe the properties of its optimal solutions and the *off-line route-set reduction* method. We present the pricing model and the viability analysis in Section IV. In Section V, we discuss the supply location problem and analyze maximum possible profits with a given total supply for a given demand configuration. We also show the equivalence with the supply optimized feed-out in this section. We present the simulation results in Section VI and provide concluding remarks in Section VII. We present the proofs of all theorems in the main text and all other propositions and lemmas in the appendices.

5) *Notation:* We use \mathbb{Z} , \mathbb{N} and \mathbb{N}_0 for the set of integers, natural numbers and $\mathbb{N} \cup \{0\}$, respectively. We use $[a, b]_{\mathbb{Z}}$ and $(a, b)_{\mathbb{Z}}$ to denote $[a, b] \cap \mathbb{Z}$ and $(a, b) \cap \mathbb{Z}$, respectively. For a compilation of most commonly used symbols please refer to Appendix A.

II. PROBLEM SETUP

In this section, we setup the coordinated first and last mode transportation problem using a macroscopic formulation. Since this is a network flow problem, we first present a graph model, and then define the route sets used for optimization. Finally, we present the optimization model.

A. Network Setup

We assume that the operator wants to provide service in a region of interest, which we model as a graph $G := (V, E)$, where V is the set of nodes and E is the set of edges between the nodes. Here, each node $l \in V$ represents an area within the region of interest. For both the feed-in and feed-out problems we have a special node, which we call as the *hub*, denoted by H . In the case of a feed-in problem we have a demand flow from the areas of the region directed to the hub H , while the opposite happens for the feed-out problem. Thus, in both these problems, we assume that for each node l there is a demand $d_l \geq 0$ that needs to be serviced. This is either spatially distributed in the region for the feed-in problem, or is concentrated at the hub H for the feed-out problem. Furthermore, we assume that the operator has a supply of $S_l \geq 0$ stationed at each node l .

If there exists a direct connection from a node $l \in V$ to a node $k \in V$, then we have a directed edge $(l, k) \in E$. Thus the edge-set, E , represents the set of all direct connections within the service region. Each edge has two weights ρ_{lk} and t_{lk} , which represent the per-unit flow travel cost and travel time, respectively. We assume that the flows in our model do not affect the congestion and thereby we assume that ρ_{lk}, t_{lk} are not affected by the flows we define later on.

We have the following assumptions for the supply configuration for both the problems.

- (A1) For the feed-in problem, we assume that $\{S_l\}_{l \in V}$ is not necessarily optimized to service the demand.
- (A2) The demand configuration $\{d_l\}_{l \in V}$ is fixed. Also, $d_l > 0$ for each node $l \neq H$ with $d_H = 0$, in both problems.
- (A3) 1 unit demand is serviced by 1 unit supply.

Assumption (A1), reflects a situation where the operator does not have enough time to move the supply to demand-specific optimal locations. These situations can arise specially in cases of emergencies, where the situation demands immediate actions from the operator. As an example, consider an emergency evacuation procedure in case of an impending natural disaster. In (A2), we make the assumption that for the nodes $l \neq H$ for which $d_l = 0$ purely for ease of exposition and there is no loss of generality and does not affect the analysis in any significant way. The assumption that $d_H = 0$ is justified because the time and cost are ‘close’ to zero for travelling within the node H . Assumption (A3) can be justified by the fact that a unit of demand and a unit of supply can refer to entities of different scales. As an example a unit demand can be 100 parcels in a courier service whereas a unit supply can be a single truck.

B. Feasible Route Sets

We define a route $r := (V_r, E_r)$ as a *walk* in the graph G . For a route r , V_r is the sequence of nodes along the route, with $V_r(j)$ being the j^{th} node on the route r , whereas $E_r(j)$ is the j^{th} edge on the route r . Therefore,

$$V_r : [1, n_r]_{\mathbb{Z}} \rightarrow V, \text{ s.t. } (V_r(j), V_r(j+1)) \in E, \quad (1)$$

$$E_r : [1, n_r - 1]_{\mathbb{Z}} \rightarrow E, \text{ s.t. } E_r(j) = (V_r(j), V_r(j+1)), \quad (2)$$

where n_r is the number of nodes (possibly repeated) on the route r . We label the *origin* and the *destination* node of the route r as $O_r \in V$ and $D_r \in V$, respectively. That is $O_r = V_r(1)$ and $D_r = V_r(n_r)$. We also define $t_r := \sum_{(l,k) \in E_r} t_{lk}$ as the *route travel time*.

We partition each route into *legs*, which are sub-routes that either originate and/or terminate at the hub H . We denote the i^{th} leg of route r by $r^i = (V_r^i, E_r^i)$ with V_r^i and E_r^i defined in the same manner as V_r and E_r respectively. We classify all legs into two types. A *circular leg* is one that originates and terminates at the hub H , while a *simple leg* is one in which exactly one of the origin, O_r , or the destination, D_r , is the hub. We identify the number of legs in a route r by $\theta_r \in \mathbb{N}$. If $\theta_r = 1$ then we say r is a *simple route*. We define a *cycle* in a leg of a route as any sub-sequence of nodes in V_r^i which starts and ends at the same node. Considering routes with multiple legs is particularly useful when the supply is not capable of meeting the demand in one go, in which case feeders can service the demand with multiple stops at the hub H .

The operator has an objective of completing the trips of the feeder vehicles within a hard time window of length T . Thus, we need to identify the routes that are feasible for each specific problem under consideration. We present here the feasible route sets for the *one-shot feed-in* and the *one-shot feed-out* problems. In the feed-in problem, the hub H is the sink of all the demand flows. Thus, a route r is feasible in the feed-in problem if its destination is $D_r = H$ and travel time $t_r \leq T$. We define the feasible feed-in route set \mathbf{R}_I as

$$\mathbf{R}_I := \{r \mid D_r = H, t_r \leq T\}. \quad (3)$$

Note that there exists a feasible route that passes through a node l if and only if there is a feasible route r with $O_r = l$. In general, there may be nodes through which no feasible route passes and can be removed from G without loss of generality.

For the feed-out problem the source of all demand is H . Therefore for a route r to be feasible for the feed-out problem it should pass through the hub. Thus, the feasible feed-out route set \mathbf{R}_O is

$$\mathbf{R}_O := \hat{\mathbf{R}} \cup \{r \mid O_r \neq H, t_r \leq T, \theta_r \geq 2\}, \quad (4)$$

$$\hat{\mathbf{R}} := \{r \mid O_r = H, t_r \leq T\}. \quad (5)$$

Here, $\hat{\mathbf{R}}$ denotes the set of all routes that originate from the hub H and have a route travel time less than or equal to T . Thus, in the feed-out problem, we may have multi-origin multi-destination routing. Since we want to consider both feed-in and feed-out problems in a unified framework, we use the \mathbf{R} as a general notation for \mathbf{R}_I or \mathbf{R}_O whenever the discussion is not particular to only one of the problems. Finally, we define $\sigma(l, k, \mathbf{R})$ as the set of all simple routes $r \in \mathbf{R}$ that originate at node l and terminate at k . Thus,

$$\sigma(l, k, \mathbf{R}) := \{r \in \mathbf{R} \mid O_r = l, D_r = k, \theta_r = 1\}. \quad (6)$$

C. Decision Variables and Constraints

We now introduce the decision variables and the constraints of the problem. For each $r \in \mathbf{R}$, f_r represents the *volume of feeders taking a route r* . Note that the route set \mathbf{R} is an exhaustive set of all feasible routes given the time window T . Therefore, if a route $r \in \mathbf{R}$ makes multiple visits to the Hub H then the route set \mathbf{R} also contains a route corresponding to every subroute of r as well as each route formed by retaining the first leg of route r and permuting its secondary legs. Thus, we assume that all the feeders in the flow f_r traverse the full route r . We let a *service tuple* (r, i, l) denote a service on node l in the i^{th} leg of route r . We let $f_r^i(l)$ denote the *volume of demand the operator serves on (r, i, l)* . We call $f_r^i(l)$ *service allocation* in short. In the feed-in problem, $f_r^i(l)$ is the demand that is picked up at (r, i, l) while in the feed-out problem it is the demand that is dropped off at (r, i, l) . We let F_l be the *total demand served on a node l* . We say a service (r, i, l) is infeasible if no demand can be serviced at node l on leg i of route r . Even if a route $r \in \mathbf{R}$ there may be some service tuples (r, i, l) on the route r that may not be feasible. For example, consider a route not originating from H in the feed-out problem. The first leg does not have any feasible services as demand can be picked-up only after the vehicles first reach H . Thus, we let $\mathbf{W}(\mathbf{R})$ be the *set of feasible service tuples given a route set \mathbf{R}* .

$$f_r^i(l) = 0, \quad \forall (r, i, l) \notin \mathbf{W}(\mathbf{R}). \quad (7)$$

This implies that no service is possible on (r, i, l) and thus all such tuples are not considered as decision variables.

We now present the constraints on the decision variables, which are as follows

$$F_l := \sum_{r \mid l \in V_r} \sum_{i \mid l \in V_r^i} f_r^i(l) \leq d_l, \quad \forall l \in V \quad (8a)$$

$$\sum_{l \in V_r^i} f_r^i(l) \leq f_r, \quad \forall i \in [1, \theta_r]_{\mathbb{Z}}, \forall r \in \mathbf{R} \quad (8b)$$

$$\sum_{r \mid O_r = l} f_r \leq S_l, \quad \forall l \in V. \quad (8c)$$

Constraint (8a) ensures that the total demand served at each node l is at most the total demand at the node. Constraint (8b) is the *leg allocation constraint*, which ensures that the sum of all allocations in a leg i on a route r is at most f_r , the feeder volume on that route, while (8c) is the *supply constraint*, which ensures that the sum of feeder volumes on all routes originating from node l , is at most the supply S_l . Note that the flow conservation constraint is automatically satisfied as volumes on each route are defined separately. Hence we do not need to consider the constraint explicitly.

D. Optimization Model

Next we give a model for the revenues and the cost to the operator and then we summarize the overall optimization problem from the operator's point of view. We let $p_r^i(l)$ denote the price charged per-unit allocation for service (r, i, l) . In general the prices $p_r^i(l)$ are dependent on many possible factors like service-times, demands etc. However,

we initially assume that the prices are independent of the optimization variables $f_r^i(l)$ and f_r as well as demand and supply variables d_l and S_l . In Section IV, we discuss the details of the pricing model and how it may be adapted for different applications. Since the prices are known, the revenue over all services is

$$\sum_{(r,i,l)} p_r^i(l) f_r^i(l).$$

We consider two different types of costs incurred by the service-provider. First, we let the travel cost for the volume of vehicles that take the route r be $c_r f_r$, which is the product of travel cost per-unit flow and the volume of vehicles that go on route r . The cost per-unit flow, $c_r > 0$, on a route r is

$$c_r := \sum_{(l,k) \in E_r} \rho_{lk} = \sum_{i=1}^{\theta_r} \sum_{(l,k) \in E_r^i} \rho_{lk} =: \sum_{i=1}^{\theta_r} c_r^i,$$

where $c_r^i > 0$ denotes the per-unit traversal cost on the i^{th} leg of route r . Second, we consider the *operational costs* (which may include incentives or commissions to the drivers and maintenance costs). We assume the operational cost is k_l units for every unit of allocation on a node l . The value of k_l can be positive or negative depending on the problem.

For each of the feed-in and feed-out problems with f_r and $f_r^i(l)$ as decision variables, we let the general *one-shot operator profit maximization problem* be

$$\begin{aligned} \max_{f_r, f_r^i(l)} J := & \sum_{(r,i,l) \in \mathbf{W}(\mathbf{R})} \beta_r^i(l) f_r^i(l) - \sum_{r \in \mathbf{R}} f_r c_r \\ \text{s.t. (8), } & f_r, f_r^i(l) \geq 0, \forall (r, i, l) \in \mathbf{W}(\mathbf{R}), \end{aligned} \quad (9)$$

where, $\mathbf{R} = \mathbf{R}_I$, for feed-in problem

$\mathbf{R} = \mathbf{R}_O$, for feed-out problem ,

and where $\beta_r^i(l)$ is the per-unit *operator income* for the service (r, i, l) , which we define as

$$\beta_r^i(l) := p_r^i(l) - k_l. \quad (10)$$

Remark II.1. (*Operational costs and profits*). A positive value of k_l is useful when the operator wants to reduce the incentives to the drivers or other maintenance costs. A negative value of k_l models situations when the operator seeks to incentivize greater servicing of the demand or when the goal is to maximize social welfare rather than monetary profits. This latter case is useful in applications such as evacuation during natural disasters, wherein the operator needs to evacuate as many people as possible, which implies that the value $\sum_l (d_l - F_l)$ should be as low as possible. This is possible with $k_l < 0$. Thus, we use the word *profit* to mean either the monetary profit or social profit or welfare. •

Thus, starting with the formulation (9) we solve three problems in this paper. First we present an offline method for reducing the size of the linear program, and thereby the computational complexity. Then, we consider the *feed-in supply optimization problem*, which extends (9) by considering the supply configuration as an optimization variable in the feed-in problem. Using this, we calculate the maximum profits for a given demand configuration. We then show that a *supply optimized feed-out* is equivalent to feed-in supply-optimization problem. Finally, we present a simple model for generating the parameters $\beta_r^i(l)$ suitable for two major scenarios. In the first scenario, the operator wants to maximize monetary profits. The second scenario prioritizes the demand serviced over monetary costs. Thus, in the latter scenario, the objective is to maximize a measure of social welfare.

III. PROPERTIES OF OPTIMAL SOLUTIONS AND OFF-LINE ROUTE ELIMINATION

In this section, we discuss some properties of the optimal solutions of the problem (9). With these properties we reduce the size of the problem by eliminating routes in a given feasible route set, \mathbf{R} , that would never be used in an optimal solution irrespective of the demand and supply configurations.

A. Properties of Optimal Solutions

We first present necessary conditions for a route to have non-zero allocations in an optimal solution of *one-shot operator profit maximization problem* (9). We derive these necessary conditions from the K.K.T. conditions for (9).

Proposition III.1. (*Necessary conditions for a route to be used in an optimal solution*). Let the route-set \mathbf{R} be either \mathbf{R}_I (feed-in) or \mathbf{R}_O (feed-out). In an optimal solution of the general one-shot problem (9), if $f_r^* > 0$ for $r \in \mathbf{R}$ then the following necessarily hold.

- (a) $f_r^i(l)^* > 0$ for some $i \in [1, \theta_r]_{\mathbb{Z}}$ and $l \in V_r^i$. Further, for each (r, i, l) , if $f_r^i(l)^* > 0$ then $\beta_r^i(l) \geq 0$.
 (b) For a circular leg i in r , the total service allocation is equal to the feeder volume on that route, f_r , that is,

$$\sum_{l \in V_r^i} f_r^i(l)^* = f_r^*, \quad \forall i \text{ such that } O_r^i = D_r^i = H. \quad (11)$$

- (c) The route r as a whole does not make a loss, that is,

$$\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \beta_r^i(l) \geq f_r^* c_r.$$

- (d) For each $i \in [2, \theta_r]_{\mathbb{Z}}$ and for $i = 1$ if leg 1 of route r is circular, there must exist an $l \in V_r^i$ such that $f_r^i(l)^* > 0$ and $\beta_r^i(l) \geq c_r^i$.
 (e) If r is a simple route ($\theta_r = 1$) then there must exist atleast one $l \in V_r$ such that $f_r^1(l)^* > 0$ and $\beta_r^1(l) \geq c_r$. \square

Proposition III.1, which we prove in Appendix B1, states that irrespective of the supply and demand configurations, if a route is used in an optimal solution then the following must hold for that route.

- There is a positive service allocation that returns non-negative operator income.
- For feeders moving in a circular leg, the optimal solution does not have an under-utilized feeder flow as it incurs a cost without earning any revenue.
- The route, as a whole, does not make a loss.
- Every secondary leg and the primary leg, if it is circular, of the route does not make a loss, that is, operator income from service allocations in such legs is no less than the per-unit traversal cost of the leg itself.
- If the route is simple then it must have at least one node with non-negative per-unit operator income.

Using Proposition III.1, we formulate an off-line route reduction method in the next subsection.

B. Offline Route Elimination

This subsection presents the *reduced route set* for the feeding problem (9) for a given route set \mathbf{R} , which may be equal to either \mathbf{R}_I or \mathbf{R}_O . In general, the feasible route-set \mathbf{R} grows rapidly with the time window T . As a result, the problem size also grows accordingly. Thus, in what follows, we obtain a reduced route-set that is formed by pruning out routes and the corresponding optimization variables that would have a zero allocation in every optimal solution under every possible supply and demand configurations. We obtain this by application of the individual properties in Proposition III.1, after eliminating the dependence on f_r and $f_r^i(l)$. We first define the reduced route set $\mathcal{R}(\mathbf{R})$, and then show that every route r such that $f_r^* > 0$ in some optimal solution for some supply and demand configuration to the feeding problem (9) belongs to $\mathcal{R}(\mathbf{R})$. We define

$$\mathcal{R}(\mathbf{R}) := \mathbf{R}_1 \cup \mathbf{R}_2 \quad (12a)$$

$$\mathbf{R}_1 := \{r \in \mathbf{R} \mid \theta_r = 1, w_r^1 \geq 0\} \quad (12b)$$

$$\mathbf{R}_2 := \{r \in \mathbf{R} \mid \theta_r > 1, \sum_{i=1}^{\theta_r} w_r^i \geq 0, w_r^i \geq 0, \forall i > 1\}, \quad (12c)$$

where

$$w_r^i := \max_{l \in V_r^i} \{\max\{\beta_r^i(l), 0\}\} - c_r^i.$$

Here, w_r^i represents the maximum profits that can be earned per unit service allocation in a leg i of route r . Thus a leg of a route is profitable on its own only if $w_r^i \geq 0$. Therefore, $\mathcal{R}(\mathbf{R})$ is the set of routes that are profitable. Intuitively one expects that only the profitable routes to be used in any optimal solution. In the following theorem we prove this intuition formally.

Theorem III.2. (Optimal solutions to the feed-in problem use only the routes from the reduced route set). For the optimization problem (9), every optimal solution for every demand and supply configuration is guaranteed to have $f_r^* = 0$ and consequently $f_r^i(l)^* = 0$ over all legs i of r , $\forall r \notin \mathcal{R}(\mathbf{R})$.

Proof. We prove this result by contradiction - hence let there exist an optimal solution such that $f_r^* > 0$ for some $r \notin \mathcal{R}(\mathbf{R})$. If $\theta_r = 1$, then it satisfies Proposition III.1(e) and as a consequence $\exists l \in V_r^1$ s.t. $\beta_r^1(l) \geq c_r = c_r^1$, which implies $w_r^1 \geq 0$. Therefore $r \in \mathbf{R}_1$. Now, if $\theta_r > 1$, then r satisfies Proposition III.1(d). Hence

$$w_r^i = \max_{l \in V_r^i} \{\max\{\beta_r^i(l), 0\}\} - c_r^i \geq 0, \quad \forall i > 1.$$

Further, r must also satisfy Propositions III.1(a) and III.1(c), that is,

$$\begin{aligned} f_r^* c_r &\leq \sum_{i=1}^{\theta_r} f_r^i(l)^* \beta_r^i(l) \leq \sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} \max\{\beta_r^i(l), 0\} f_r^i(l)^* \\ &\leq \sum_{i=1}^{\theta_r} \max_{l \in V_r^i} \{\max\{\beta_r^i(l)\}, 0\} f_r^*, \end{aligned}$$

where we have used the fact that $f_r^i(l)^* > 0$ only if $\beta_r^i(l) \geq 0$ for the second inequality and the third inequality follows from (8b). Hence, r must satisfy $\sum_{i=1}^{\theta_r} w_r^i \geq 0$ which implies $r \in \mathbf{R}_2$. Therefore, in either case, $r \in \mathbf{R}_1 \cup \mathbf{R}_2 = \mathcal{R}(\mathbf{R})$. This is a contradiction. \square

In the proof we don't explicitly check Proposition III.1(a) for the route but one can verify that $\forall r \in \mathcal{R}(\mathbf{R})$ there exists $\beta_r^i(l) \geq 0$ where $f_r^i(l)^* > 0$ is possible for some supply and demand configuration. Thus, replacing \mathbf{R} with $\mathcal{R}(\mathbf{R})$ in Problem (9) causes no *approximation* or *loss* of any optimal solutions.

IV. PRICING MODEL AND VIABILITY ANALYSIS

In this section, we propose a simple pricing model for the one-shot problems based on the travel times and costs. This model helps in further analysis of the one-shot model, like calculating maximum profits presented in next section. We first present a general pricing model and explore the question of viability of the feeder service. Then, we specialize the general pricing model for two cases - for maximization of serviced demand and for maximization of operator profits. The former case is applicable in scenarios like evacuation or relief supply at the time of disasters. The latter case is applicable for large sporting and other public events as well as for one-shot courier service.

A. General Pricing Model

We utilize the concept of *value of time* (V.o.T.), which associates a monetary cost to the travel times. In particular, we let α be the monetary value of unit time. Then, the *perceived cost* is $M + \alpha\tau$ for a transportation service that takes τ units of time and charges a monetary price M . In this paper, we assume that the V.o.T. is the same for all the nodes. However, our model and results are easily generalizable to the case where the V.o.T. is node-dependent.

For each node $l \in V$, we let $g_l := \alpha\eta_l + \zeta_l$ be the perceived cost for the *best alternate transport*, which has a travel time η_l and has a price of ζ_l . For the service (r, i, l) to be viable, the perceived cost of the feeder service should be less than or equal to perceived cost for the best alternate transportation at node l , that is

$$p_r^i(l) + \alpha t_r^i(l) \leq \zeta_l + \alpha\eta_l =: g_l,$$

where $t_r^i(l)$ is the *service time* for the service tuple (r, i, l) . This represents the duration of service including the waiting time and the actual travel time for the service tuple (r, i, l) .

Referring to Remark II.1, note that since the motivation of the operator is to maximize profits, either social or monetary, the value of $p_r^i(l)$ for any service (r, i, l) has to be as high as possible. Thus, the optimal prices for the service (r, i, l) is

$$p_r^i(l)^* = \zeta_l + \alpha\eta_l - \alpha t_r^i(l) = g_l - \alpha t_r^i(l). \quad (13)$$

The next challenge is modelling of g_l , the perceived costs of the best alternate transportation. We discuss this next, following which we explore the question of economic viability of the feeder service.

1) *Modelling the Perceived Costs of the Best Alternate Transportation:* In order to systematically generate g_l for each node $l \in V$, we first assume that the best alternate transport available in the region costs bc_r and takes time t_r along the route $r \in \mathbf{R}$. The *cost-factor* $b \geq 0$ signifies the cost to a passenger of the best alternate transportation relative to the feeder service. For simplicity, we assume it to be the same through out the service area. We do this purely to simplify the notation and to carry out some analysis regarding the economic viability of the feeder service.

The rest of the model and analysis in the paper can easily handle more complicated perceived costs for the best alternate transport. First, we let

$$r(O, D, b)^* \in \underset{r \in \sigma(O, D, \mathbf{R})}{\operatorname{argmin}} \{ \alpha t_r + bc_r \} \quad (14)$$

where $r(O, D, b)^*$ is a route that the best alternate transport uses between a given origin node O and the destination node D . Note that $\mathbf{R} = \mathbf{R}_I$ or \mathbf{R}_O depending on the problem and we are interested in setting one of O or D to the hub node H and the other to l , a node of interest. However, in order to avoid cumbersome notation, we use the shortened notation of $r(l, b)^*$ as defined in Table I. With this notation, we define the perceived cost of the best

Shorthand Notation	Complete Notation for Given \mathbf{R}	
	\mathbf{R}_I	\mathbf{R}_O
(Origin, Destination) = (O, D)	(l, H)	(H, l)
$r(l, b)^*$	$r(l, H, b)^*$	$r(H, l, b)^*$
$c_{r(l, b)^*}$	$c_{r(l, H, b)^*}$	$c_{r(H, l, b)^*}$
$t_{r(l, b)^*}$	$t_{r(l, H, b)^*}$	$t_{r(H, l, b)^*}$

TABLE I: Shorthand notations of route parameters for $\mathbf{R} = \mathbf{R}_I$ or \mathbf{R}_O

alternate transport between nodes l and H given the route set \mathbf{R} as

$$g_l(b) := \alpha \eta_l + \zeta_l, \quad \eta_l = t_{r(l, b)^*}, \quad \zeta_l = bc_{r(l, b)^*}. \quad (15)$$

Here, η_l , ζ_l and $g_l(b)$ are the travel time, cost and the perceived cost of the best alternate transport between l to node H with (O, D) defined as per Table I based on \mathbf{R} .

Remark IV.1. (*Effect of the cost-factor on best alternate transportation*). For a fixed T and given $\mathbf{R} = \mathbf{R}_I$ or \mathbf{R}_O , as there are finitely many routes, the perceived cost $g_l(b)$ is a piecewise-linear, increasing, concave function of b for each node $l \in V$. Further, at b , the slope of $g_l(b)$ is equal to $c_{r(l, b)^*}$ and the g intercept is $t_{r(l, b)^*}$. Thus, η_l and ζ_l are also unique for each b except where the slope of $g_l(b)$ changes. For $b = 0$ and $b = \infty$, the routes $r(l, b)^*$ are the fastest and cheapest, respectively. For any b , $r(l, b)^*$ is such that $t_{r(l, b)^*} \geq t_{r(l, 0)^*}$ and $c_{r(l, b)^*} \geq c_{r(l, \infty)^*}$. •

2) *Viability of Feeder Services:* We first present conditions on the value of b for the reduced route set $\mathcal{R}(\mathbf{R})$ to be non-empty and as a result for the feeder service to be viable. An important factor for determining viability is the operational cost/revenue k_l . As stated earlier, k_l represents the cost that operator pays as commissions or represents the incentive/subsidy for serving the node l . Thus, next we discuss the viability of service depending on the sign of k_l .

Lemma IV.2. (*Conditions on b for viability of feeder service*). The reduced route set $\mathcal{R}(\mathbf{R})$ is non-empty if and only if $g_l(b) \geq g_l(1) + k_l$, $l \in V$, where $g_l(b)$ is given by (15). Further,

- (a) If $k_l > 0 \forall l \in V$ then $\mathcal{R}(\mathbf{R})$ is non-empty only if $b > 1$.
- (b) If $k_l < 0$, for all $l \neq H$ and $b = 1$, then $\mathcal{R}(\mathbf{R}) \neq \emptyset$.

We present the proof in Appendix C1. Next, we present necessary conditions for the viability of *multi-service routes*. These are routes with multiple services over different legs, viability of which helps service more demand with lesser supply.

Proposition IV.3. (*Necessary value of b for multi-service viability*). Consider the graph G , the route set $\mathbf{R} \in \{\mathbf{R}_I, \mathbf{R}_O\}$ and the corresponding (O, D) defined in Table I. Suppose $k_H > 0$ and let $c^*(p, q)$ be the cheapest travel cost from p to q for route set \mathbf{R} . Suppose that $\exists(r, i_1, l_1) \in \mathbf{W}(\mathbf{R})$ and $\exists(r, i_2, l_2) \in \mathbf{W}(\mathbf{R})$ for a route $r \in \mathcal{R}(\mathbf{R})$, two distinct legs, $i_1, i_2 \in [1, \theta_r]_{\mathbb{Z}}$ and $l_1, l_2 \in V_r$, where $\mathbf{W}(\mathbf{R})$ is defined as per (7). Then,

- (a) $g_l(b) \geq g_l(1) + c^*(D, O) + k_l$, for some $l \in V$, $l \neq H$
 (b) $b \geq \hat{b}_l^*$ for some $l \in V$, $l \neq H$, where $\hat{b}_l^* := \left(\frac{k_l + c^*(D, O) + \alpha(t_{r(l,1)^*} - t_{r(l,\infty)^*}) + c_{r(l,1)^*}}{c_{r(l,0)^*}} \right)$
 (c) If $k_l \geq 0$ for all $l \in V$ then $\exists l \in V$, $l \neq H$ such that $b \geq b_l^*$, where $b_l^* := \left(1 + \frac{k_l + c^*(D, O) + \alpha(t_{r(l,1)^*} - t_{r(l,\infty)^*})}{c_{r(l,1)^*}} \right)$.

Also, b calculated by (a) is greater than \hat{b}_l^* . Also, $b_l^* \geq \hat{b}_l^*$.

The proof is given in Appendix C2. Note that the assumption that $k_H > 0$ causes no loss of generality in our problem. This is because in both the feed-in and feed-out problems, we never have demand that wants to go from H to H . So, we can choose $k_H > 0$, without affecting any optimization variables.

With Proposition (IV.3), given the graph G , the value of time α and the value of b , the operator can evaluate *multi-service* viability of the feeder service. Also, given a node l , Proposition IV.3(a) can be interpreted as a necessary condition for viability of a single depot V.R.P. service. Note that we give three necessary conditions in Proposition IV.3. The bound in claim (a) requires the computation of $g_l(b)$ for each b . In comparison, condition (b) is a computationally simpler relation but provides a bound lower than the one in condition (a). Condition (c) provides a stricter bound than condition (b) under the assumption that $k_l > 0$ for all $l \in V$.

Remark IV.4. (*Reduced route set reaches a constant as the destination time T is increased*). There exists a time T^* such that for all $T \geq T^*$, the set $\mathcal{R}(\mathbf{R})$ is the same. This is because even though the set \mathbf{R} includes more and more routes as T increases, for a route to be in $\mathcal{R}(\mathbf{R})$, the service times $t_r^i(l)$ cannot exceed a certain value while satisfying $\beta_r^i(l) \geq 0$. Furthermore, even for services with lower service times, higher costs would render longer routes unprofitable. Thus $\mathcal{R}(\mathbf{R})$ do not have such routes. This is particularly useful as the set \mathbf{R} and problem (9) keeps growing with T , whereas the size of (9) with $\mathcal{R}(\mathbf{R})$ instead of \mathbf{R} does not grow for $T \geq T^*$. •

B. Pricing Models for Different Scenarios

We next demonstrate the generality of our model and explain the interpretations of $\beta_r^i(l)$, $t_r^i(l)$, k_l and thereby the objective of (9) for two broad scenarios.

1) *Demand Maximization*: In the first scenario, the operator aims to maximize the social welfare by maximizing the demand that is serviced. This case is applicable in disaster management scenarios, where the goal is to maximize the evacuation (feed-in) or relief efforts (feed-out). Thus in this scenario, one could let the service time $t_r^i(l)$ for (r, i, l) be the drop-off time with respect to a suitable reference initial time. In other words, the service time can include both the time to wait for the feeders to pick-up as well as the travel time from node l to the drop-off location on leg i of route r .

Next, k_l in this scenario can represent a sense of priority with which a region l must be serviced. Referring to Remark II.1, we thus let k_l 's to be negative. For example, regions with greater danger of loss of life could have a more negative value of k_l . Alternatively, k_l could also represent monetary incentives or subsidies that the operator would receive from, say the government, for servicing the region l . The ‘‘operator income’’ $\beta_r^i(l)$ in this scenario may represent the ‘‘social revenues’’ rather than monetary income for a service (r, i, l) . Alternatively, $\beta_r^i(l)$ could also represent monetary income if k_l represents monetary incentives or subsidies that are given to the operator.

Thus in this scenario, the objective of (9) is to maximize social welfare in a cost efficient and coordinated manner. Furthermore, Lemma IV.2 and Proposition IV.3 provide conditions when relief efforts are not viable either due to lack of time before the disaster or due to the costs.

2) *Profit Maximization*: In this scenario, the operator seeks to maximize the monetary profits. Thus, $t_r^i(l)$ may or may not include wait times depending on the specific instance of the problem. The parameter k_l could be the fixed costs that the operator needs to pay for all the expenses such as maintenance of the fleet, commissions to the drivers etc. In this scenario, it is also possible that a region l may pay incentives for availing service from the operator. Therefore, k_l can potentially be negative though in general it is positive for this scenario. The objective in (9) therefore represents the monetary profits. This scenario is applicable for coordinated transportation services in the context of less frequent events like long distance courier services, where holding at the main airport hub is costly and parcels need to arrive and depart just in time; or large social gatherings or sporting events. Further, the operator can also prioritize certain nodes using either k_l 's or by choosing an appropriate node-dependent V.o.T.

V. FEED-IN SUPPLY OPTIMIZATION PROBLEM AND EQUIVALENCE TO ONE-SHOT FEED-OUT

Until the previous section, we discussed the one-shot feeding problem, given a supply and demand configuration. However, an operator may also be interested in determining the best possible supply configuration for a demand

configuration so that profits or social welfare are maximized. In the scenarios where evacuation or relief efforts in the context of a disaster are the goal, predictive models of the impact of a disaster may be available. This information could be used to pre-position the service vehicles well before the evacuation or relief distribution begins. In this section, we first discuss the problem of supply optimization for feed-in. Then, we discuss supply optimization for feed-out and show its “equivalence” to supply optimized feed-in under mild assumptions.

We assume that the demand configuration and the total available supply s are given. We let the supply at the nodes $\{S_l\}_{l \in V}$ also be optimization variables. Then, *feed-in supply optimization problem* can be stated as

$$\begin{aligned} \max_{S_l, f_r, f_r^i(l)} \bar{J} &:= \sum_{(r,i,l)} \beta_r^i(l) f_r^i(l) - \sum_{r \in \mathbf{R}_I} f_r c_r \\ \text{s.t. (8), } \sum_{l \in V} S_l &\leq s, \quad \forall l \in V_r \\ f_r, f_r^i(l), S_l &\geq 0, \quad \forall r \in \mathbf{R}_I, \quad \forall i \in [1, \theta_r]_{\mathbb{Z}}, \quad \forall l \in V_r. \end{aligned} \quad (16)$$

For analysing the maximum profit as a function of the total supply s , we make the following assumptions.

(A4) For each node l in the graph $\exists r \in \mathbf{R}_I$ s.t. $O_r = l$, $\theta_r = 1$ and $\beta_r^1(l) - c_r^1 \geq 0$.

Note that there is no loss of generality in Assumption **(A4)**. We elaborate on this point in the sequel.

A. General Properties of Feed-in Supply Optimization

In the following proposition, we give some properties of the optimal solutions of the *feed-in supply optimization problem* (16). From the result, we can also reason that there is no loss of generality in the Assumption **(A4)**. We present the proof of the result in Appendix D1.

Proposition V.1. (*Properties of optimal supply configurations and allocations*). *In every optimal solution to the problem (16), the following hold:*

- (a) If $f_r^* > 0$ then $r \in \mathcal{R}(\mathbf{R}_I)$
- (b) For each $r \in \mathcal{R}(\mathbf{R}_I)$, $f_r^1(O_r)^* = f_r^*$. Consequently,

$$\sum_{l \in V_r^i} f_r^i(l)^* = f_r^*, \quad \forall i \in [1, \theta_r]_{\mathbb{Z}}, \quad \forall r \in \mathcal{R}(\mathbf{R}_I), \quad (17)$$

and no route originating from H is used.

- (c) For a route $r \in \mathcal{R}(\mathbf{R}_I)$ if $f_r^* > 0$ then $\beta_r^1(O_r) \geq c_r^1$. Consequently, $\forall i \in [1, \theta_r]_{\mathbb{Z}}, \exists l \in V_r^i$ such that $f_r^i(l)^* > 0$. Moreover, for (r, i, l) , $f_r^i(l)^* > 0$ only if $\beta_r^i(l) \geq c_r^i$.
- (d) If a node l does not satisfy the property in **(A4)** then $f_r^i(l)^* = 0$ for all (r, i, l) that serve the node l .
- (e) A route r with a cycle in the first leg is not used, that is $f_r^* = 0$.
- (f) If $s \leq \sum_{l \in V} d_l$ and **(A4)** holds then $\sum_{r|O_r=l} f_r^* = S_l \leq d_l, \forall l \in V$. □

From Proposition V.1 we see that there is no loss of generality in the Assumption **(A4)**. This is because if for a node l , $d_l = 0$ then, by Proposition V.1(b), all routes r originating at l have zero flow ($f_r = 0$) in every optimal solution. Similarly, Proposition V.1(d) says that in every optimal solution there is no allocation on nodes that violate Assumptions **(A4)**. With Proposition V.1(d) one can eliminate the nodes that do not follow assumption **(A4)**. Also, as a consequence of Propositions V.1(b) and V.1(c), one can eliminate the route flow variables f_r and remove routes without a *profitable* pickup at their origins. With Proposition V.1(e) we can eliminate every route with a cycle in the first leg. Thus we construct the *reduced route set* for (16), \mathbf{R}^- , as

$$\mathbf{R}^- := \{r \in \mathcal{R}(\mathbf{R}_I) | O_r \neq H, \beta_r^1(O_r) \geq c_r^1, \text{ no cycles in } r^1\}. \quad (18)$$

Further, using Proposition (V.1) and (17), we can solve (16) with equality in the constraints of (8b) and (8c). Thus, we can reduce (16) to an optimization problem over decision variables $f_r^i(l)$, the allocations, and S_l , the supply at a node. This elimination of the variables f_r leads to a significant reduction in the number of optimization variables, specifically equal to the number of routes in \mathbf{R}^- . As a result, we can express the supply optimization problem as

$$\max_{S_l, f_r^i(l)} \bar{J} = \sum_{(r,i,l) \in \mathbf{W}(\mathbf{R}^-)} (\beta_r^i(l) - c_r^i) f_r^i(l) \quad (19)$$

$$\begin{aligned} \text{s.t. } F_l &\leq d_l, \sum_{l \in V_r^i} f_r^i(l) = f_r, \sum_{r | O_r=l} f_r = S_l, \sum_{l \in V} S_l \leq s, \\ f_r^i(l), S_l &\geq 0, \forall (r, i, l) \in \mathbf{W}(\mathbf{R}^-). \end{aligned}$$

This is a simpler problem to solve for a sequence of values of s than (16). Also, as we show in the next subsection, this formulation makes the feed-out problem computationally simpler.

1) *Absolute Maximum Profits:* With the objective (19) and the discussion in Section IV, we can also analyze the absolute maximum profits an operator can earn, over all supply configurations, for a given demand configuration. Quantification of the absolute maximum profits is useful for determining profitability of the service from the operator's perspective. To arrive at the value of *absolute maximum profits*, denoted by J_{\max} from here on, we assume (without loss of generality) that supply is sufficient, i.e. $s \geq \sum_l d_l$. We denote the set of simple routes with the maximum rate of profits for a pickup at l by $\mathcal{S}(l)$. Formally,

$$\mathcal{S}(l) := \operatorname{argmax}_{r \in \sigma(l, H, \mathbf{R}_I) \cap \mathbf{R}^-} \{\beta_r^1(l) - c_r\}. \quad (20)$$

Lemma V.2. *Given pricing models (10) and (13) for each node $l \in V$, $\beta_r^1(l) - c_r > \beta_{\bar{r}}^i(l) - c_{\bar{r}}^i$ for all $r \in \mathcal{S}(l)$, $\bar{r} \notin \mathcal{S}(l)$ and $i \in [1, \theta_{\bar{r}}]_{\mathbb{Z}}$. Further, $\forall r \in \mathcal{S}(l)$ the perceived cost ($\alpha r + c_r$) is the least from node l . \square*

The lemma is proved in Appendix D2. This lemma implies that the pickups in the simple routes with maximum rate of profits are always more profitable than those of any other route. Next, we state the properties of optimal solutions of (19) for sufficient supply.

Theorem V.3. *(Properties of optimizers under sufficient supply). If Assumption (A4) holds then the following hold.*

- (a) *If $s \geq \sum d_l$ then $f_r^* = 0, \forall r \notin \mathcal{S}(O_r)$. Further, for each $l \in V \setminus \{H\}$, $F_l^* = \sum_{r \in \mathcal{S}(l)} f_r^* = d_l$ and $S_l^* \geq d_l$.*
 (b) *For any total supply s , the set of supply configurations defined in (21) always contains an optimal configuration.*

$$\left\{ \{S_l\} \mid \sum_{l \in V} S_l = s, S_H = \max\{0, s - \sum_{l \neq H} d_l\} \right\} \quad (21)$$

- (c) *The maximum profits over all supply configurations is*

$$J_{\max} = \sum_{l \in V} d_l \max_{r \in \mathcal{S}(l)} \{\beta_r^1(l) - c_r^1\}. \quad (22)$$

Proof. (a): As $s \geq \sum_l d_l$, consider a solution where, $\sum_{r \in \mathcal{S}(l)} f_r^1(l) = d_l, f_r^i(l) = 0$ for all $r \notin \mathcal{S}(l)$ for each $l \in V$,

and $S_l = \sum_{r \in \mathcal{S}(l)} f_r^1(l), \forall l \neq H$, and $S_H = s - \sum_l d_l$. One can verify that such a solution is feasible under

Assumption (A4). From Lemma V.2, we know that $\beta_r^1(l) - c_r > \beta_q^i(l) - c_q^i$ for all $r \in \mathcal{S}(l)$ and $q \notin \mathcal{S}(l)$. Then the structure of the objective function (19) implies that this solution is optimal.

(b): Part (b) follows from arguments given above and Proposition V.1(f).

(c): Given part (a), we now see that the maximum profits must satisfy (22), in which the term indexed by l corresponds to the profits from node l . \square

This theorem gives the absolute maximum profits for a given demand configuration over all supply configurations. The value of J_{\max} is easily computable with knowledge of maximum rates of profit for simple routes and demand configuration.

B. Supply Optimized Feed-out

In this subsection, we explore *supply optimized feed-out problem*. We denote the variables and parameters of this problem with a hat. Thus $\hat{f}_r, \hat{f}_r^i(l)$ and $\hat{\beta}_r^i(l), \hat{S}_l$ represent flow, allocations and operator income respectively. Thus

with demand configuration $\{\hat{d}_l\}$, and optimization variables $\hat{f}_r, \hat{f}_r^i(l)$, total allocation \hat{F}_l and supply configuration $\{\hat{S}_l\}$, the problem is

$$\begin{aligned} \max_{\hat{f}_r, \hat{f}_r^i(l), \hat{S}_l} J &:= \sum_{(r,i,l) \in \mathbf{W}(\mathbf{R}_O)} \hat{\beta}_r^i(l) \hat{f}_r^i(l) - \sum_{r \in \mathbf{R}_O} \hat{f}_r c_r \\ \text{s.t. (8),} \quad &\sum_l \hat{S}_l \leq s, \quad \hat{S}_l \geq 0, \quad \forall l \in V \\ &\hat{f}_r, \hat{f}_r^i(l) \geq 0, \quad \forall (r, i, l) \in \mathbf{W}(\mathbf{R}_O). \end{aligned} \quad (23)$$

It is trivial to note that the optimal supply configuration in the case of feed-out is to concentrate the supply at the hub H . Thus, we assume that $\hat{S}_H = s$, for total supply s . Furthermore, all the optimal routes lie in $\hat{\mathbf{R}}$ (see (5)). With this assumption we show that for the supply optimized feed-out problem, an equivalent feed-in supply optimization problem exists with the same optimization value and related optimizers. Thus, for the feed-out problem, we assume a graph $\hat{G} = (\hat{V}, \hat{E})$ and construct a solution given through the following conditions.

- (C1) For the feed-out problem we assume that $\{\hat{S}_l\}$ is such that $\hat{S}_H = s$ and $\hat{S}_l = 0$.
- (C2) Let the graph $G := (V, E)$ be such that $V := \hat{V}$, edge $(k, l) \in E$ iff $(l, k) \in \hat{E}$ and $(\rho_{kl}, t_{kl}) = (\hat{\rho}_{lk}, \hat{t}_{lk}) \forall (l, k) \in \hat{E}$. Also, we assume $T = \hat{T}$.
- (C3) The supply configuration for (16) is chosen from the supply set (21). Also, spatial demand configurations are same, i.e. $\{\hat{d}_l\} = \{d_l\}$.

Condition (C1) requires the supply to be concentrated at H , which as we stated earlier is optimal for the feed-out problem. Condition (C2) represents a “reversed” graph for the feed-out problem with same parameters. Similarly, Condition (C3) “reverses” the demand. Next, we give a mapping between the route set \mathbf{R}_I and $\hat{\mathbf{R}}$.

Remark V.4. (Equivalent Route-Sets). Given a route r on graph \hat{G} , let $\phi_{\hat{G}G}(r) := \bar{r}$, a route in G (defined by (C2)) where $\theta_r = \theta_{\bar{r}}$, $V_r(i) = V_{\bar{r}}(n_{\bar{r}} - i + 1)$ and $E_{\bar{r}}(n_{\bar{r}} - i) = (V_r(i + 1), V_r(i))$. Then, $\forall r \in \hat{\mathbf{R}}, \exists \phi_{\hat{G}G}(r) = \bar{r} \in \mathbf{R}_I$ with $c_r = c_{\bar{r}}$ and every service tuple (r, i, l) of r mapping to $(\bar{r}, \theta_r - i + 1, l)$. •

We can now show that $\hat{\beta}_r^i(l)$ and $\beta_{\bar{r}}^{\theta_r - i + 1}(l)$ are the same under Condition (C2).

Lemma V.5. For the feed-out problem and the corresponding feed-in problem, $\hat{\beta}_r^i(l) = \beta_{\bar{r}}^{\theta_r - i + 1}(l)$, where $\bar{r} = \phi_{\hat{G}G}(r)$. □

The proof of this Lemma is given in Appendix D3. Next we show equivalence of the two problems.

Theorem V.6. (Equivalence of the feed-out problem and the feed-in supply optimization problem). Under Assumption (A4), the feed-out problem defined by route set $\hat{\mathbf{R}}$ with supply concentrated at the Hub can be represented by an equivalent feed-in supply optimization problem (16) (which can be constructed as in Conditions (C1)- (C3)) with $\hat{J}^*(s) = \bar{J}^*(s)$, $\forall s \geq 0$. Further, for all optimal solutions

- (a) $\hat{f}_r^i(l)^* = f_{\bar{r}}^{\theta_r - i + 1}(l)^*$, $\forall (r, i, l)$ and $\bar{r} = \phi_{\hat{G}G}(r)$,
- (b) $\hat{f}_r^* = f_{\bar{r}}^*$, $\bar{r} = \phi_{\hat{G}G}(r)$, $\hat{F}_l^* = F_l^*$, $\forall l \in V$.

Proof. Each pair of $r \in \hat{\mathbf{R}}$ and $\phi_{\hat{G}G}(r) = \bar{r} \in \mathbf{R}_I$ satisfy the relationship given in Remark V.4. Using Lemma V.5, we see that $\hat{\beta}_r^i(l) = \beta_{\bar{r}}^{\theta_r - i + 1}(l)$ and given $T = \hat{T}$ we conclude that the cost functions are equivalent, i.e. $\bar{J} \equiv \hat{J}$ under the assumption $\hat{f}_r^i(l) = f_{\bar{r}}^{\theta_r - i + 1}(l)$. Hence, it is sufficient to prove equivalence of the constraints in both problems.

Leg Constraints: In both problems given flows \hat{f}_r and $f_{\bar{r}}$ the constraint for leg i and route r in *feed-out* problem and the constraint for leg $\theta_r - i + 1$ and route \bar{r} for the feed-in supply optimization (see (19)) are equivalent.

Demand Constraints: Constraint $F_l \leq d_l$ are equivalent under the assumption that $d_l = \hat{d}_l$.

Supply Constraints: To show this equivalence, we let $\{\hat{S}_l\}_l$ be the supply configuration at the end of *feed-out*. We know that for any route r , the flow terminates at D_r . Therefore, the final supply located at any node l is $\hat{S}_l = \sum_{r|D_r=l} \hat{f}_r$. Therefore, Constraint (8c) for feed-out can be rewritten as

$$\sum_r \hat{f}_r = \sum_l \sum_{r|D_r=l} \hat{f}_r = \sum_l \hat{S}_l \leq s,$$

which implies, $\hat{S}_l = \sum_{r|D_r=l} \hat{f}_r$, and $\sum_l \hat{S}_l \leq s$. Furthermore, one can see that $\sum_r f_r = \max\{\sum d_l, s\}$ which implies that all the supply reaches H in feed-in problem. Now, with the assumption that supply configuration for *feed-in* is chosen from the set (21), one can see that constraint (8c) along with $\sum_{l \in V} S_l \leq s$ are equivalent to ones stated above as consequences of Proposition V.1(f) for $s \leq \sum_l d_l$ and Theorem V.3(a) with Lemma V.3(b) for $s \geq \sum_l d_l$. \square

Theorem V.6 establishes the equivalence of the feed-out problem on the graph \hat{G} to the supply optimization problem on the graph G , formed using (C2). Thus, one may solve either problem and obtain a solution to the feed-out problem. More importantly, all the properties and results of supply optimization problem apply for the supply optimized feed-out problem, though in a “reversed” manner.

VI. SIMULATIONS

Since the general operator feeding problem (9) is a linear program, we utilized CVXpy [49] for modelling the problem. We also performed simulations for a microscopic formulation of the problem (9), with integer constraints on the optimization variables. We performed these microscopic simulations using Gurobi [50] and ECOS optimizers. We performed all the simulations on a 3.7 GHz i7-7th generation machine with 32 GB of RAM. We present simulation results for several graphs that include test graphs and a graph constructed from real world data. With these simulation results, we illustrate various aspects of our analytical results and comparisons with solutions of the problem in a microscopic formulation.

A. Numerical Illustration of Analytical Results

1) *Test Graphs Utilized for Simulation:* We have utilized the two test graphs in Figure 1 for numerical illustration of our analytical results. The graph in Figure 1a has undirected or bidirected edges and edge weights (travel time and travel cost). Some highway networks (such as in semi-urban or rural areas) and railway networks are of this nature. The graph in Figure 1b has directed arcs while the travel times and costs depend on the direction of the arc between each pair of nodes. Such conditions are common for transportation networks in cities.

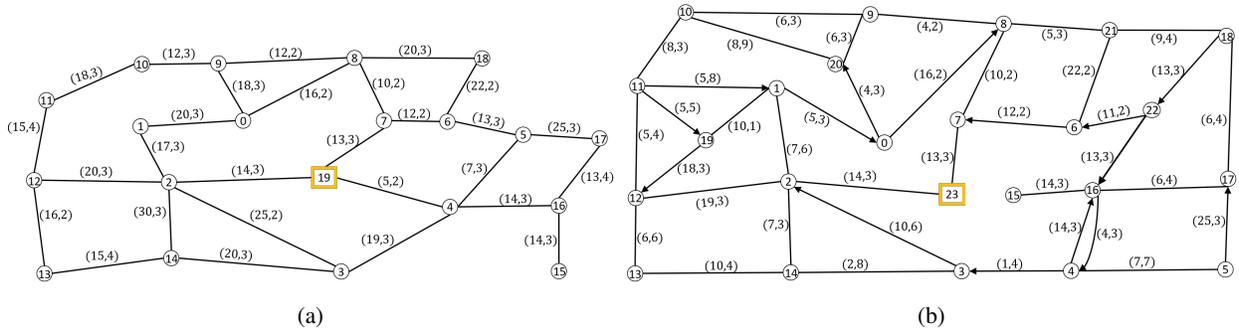


Fig. 1: Test graphs for simulations. (a) Graph I and (b) Graph II. Numbers in the circles represent node index, an arrow between nodes l and k indicates a directed arc from l to k and a line without an arrow between nodes l and k indicates a bi-directional edge. The tuple (a, b) on the arc (l, k) represents (ρ_{lk}, t_{lk}) and the Hub node H is marked in square.

2) *Variation of Route Set:* Here, we consider the feed-in problem for the two graphs in Figure 1, that is $\mathbf{R} = \mathbf{R}_I$, and study the variation in $\mathcal{R}(\mathbf{R}_I)$ as a function of the time-window, T and the cost factor, b . Given T , we utilize a brute force method to enumerate all the routes in a graph. Furthermore, with b and T , we utilize (15) and a V.o.T. of $\alpha = 0.5$ to generate the best alternate transportation time and cost, η_l , ζ_l respectively, for each node l . Then, we use (13) to generate prices for each service tuple (r, i, l) .

Figure 2 shows the variation of the number of routes in \mathbf{R}_I and $\mathcal{R}(\mathbf{R}_I)$ as a function of T for Graph I and Graph II. We see an exponentially increasing trend for the number of routes in all cases. However, as discussed in Remark IV.4, one can observe that the reduced route sets saturate eventually for large enough T . Moreover, the value of T at which this saturation occurs increases with b . Finally, even prior to saturation, we see that there is a significant reduction in the route set compared to the original route set.

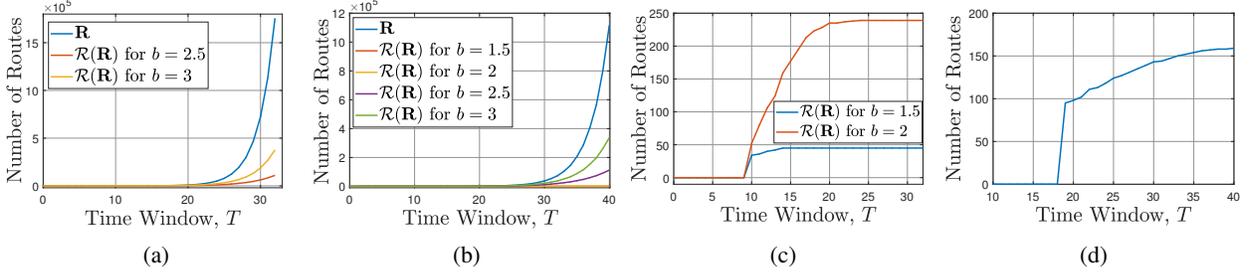


Fig. 2: (a), (b): Variation of number of routes in $\mathbf{R} = \mathbf{R}_I$ and $\mathcal{R}(\mathbf{R}_I)$ as a function of T for Graph I and Graph II, respectively. (c), (d): Saturation observed in $\mathcal{R}(\mathbf{R}_I)$ with $b = 1.5, 2$ for Graph I and with $b = 1.5$ for Graph II.

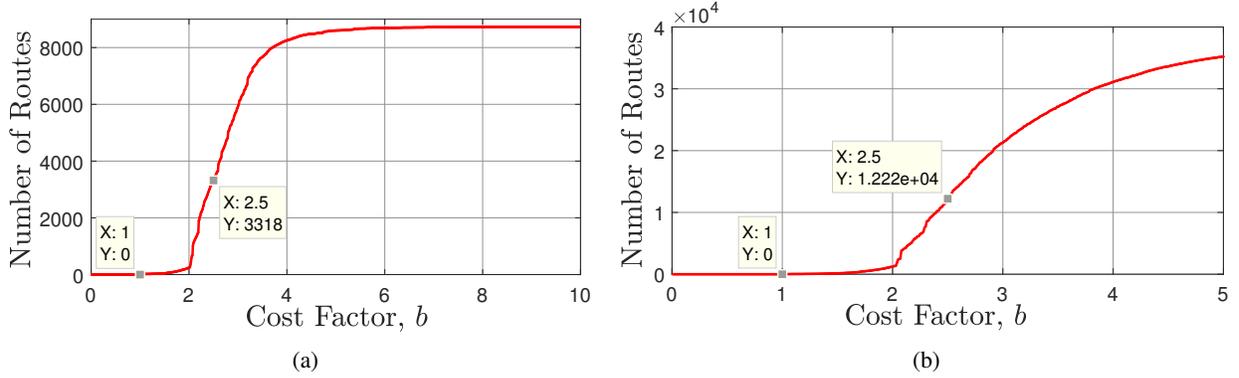


Fig. 3: Variation of number of routes in $\mathcal{R}(\mathbf{R}_I)$ versus b for $\alpha = 0.5$. Note that number of routes is 0 till $b = 1$. (a): For Graph I and $T = 20$. (b): For Graph II and $T = 30$.

Figure 3 shows the number of routes in $\mathcal{R}(\mathbf{R}_I)$ as a function of b (with a step size of 0.01) for the graphs in Figure 1. We note that the first route with origin as H and the first multi-legged route in $\mathcal{R}(\mathbf{R}_I)$ occur at $b = 1.71$. Proposition IV.3(a), IV.3(b) and IV.3(c) give necessary lower bounds on b for the existence of multi-legged routes in $\mathcal{R}(\mathbf{R}_I)$ as 1.7027, 1.411 and 1.692, respectively. As one can see, \hat{b}_i^* gives a more conservative bound compared to b_i^* . In each case, the origin of the multi-legged route is the node $l = 0$. In Figure 3, we also see that there is a significant increase in the number of routes around $b = 2.1$. This can be explained by the fact that $b_i^* \in (2, 2.1)$ for 5 nodes.

3) *Feed-In and Comparison with Single-Depot Routing Problem for Graph II:* In the rest of this subsection, we present results for Graph II. We set $b = 2.5$ and $T = 30$, for which $\mathcal{R}(\mathbf{R}_I)$ has 12219 routes and 45050 optimization variables. We also note that all nodes satisfy Assumptions (A2) and (A4). We simulated the *feed-in* problem for a fixed demand profile with d_l , for each $l \in V$, drawn uniformly from $[0, 250]_{\mathbb{Z}}$. The total demand was $\sum_l d_l = 2468$. Figure 4 shows the simulation results. We utilized Proposition V.1 to generate the route set \mathbf{R}_I^- which had 6265 routes and the resulting number of optimization variables was 12695. In Figure 4a, maximum profits for given total supply (marked in red line) is from Proposition V.1. The maximum profits as a function of total supply s converge to the absolute maximum profits, $J_{\max} = 105033.5$ (given by Theorem V.3). We also simulated an *equivalent macroscopic single-depot* V.R.P., with all supply at H , i.e. $S_H = s$, $S_l = 0$, $\forall l \neq H$. We observe in Figure 4a that the profits earned in this latter case are far lower, compared to that of any randomly chosen supply configurations. This is explained by two factors - insufficient time-window and cost-factor. Given $T = 30$, 4 nodes do not have $r \in \mathbf{R}_I$ such that $O_r = H$ and $l \in V_r$. Also given $b = 2.5$, only 16 of the 23 nodes satisfy Proposition (IV.3(a)), implying at-least 7 nodes do not have $r \in \mathcal{R}(\mathbf{R}_I)$ with $O_r = H$ and $l \in V_r$. The necessary value of b is 3.06 for all nodes to satisfy Proposition IV.3(a).

In Figure 4b, we also see the number of routes used to generate maximum profits generally increases with s though after a point the number of routes used starts to reduce. We imposed the added restriction that the supply configuration is chosen from the set (21) to compare with the equivalent feed-out problem.

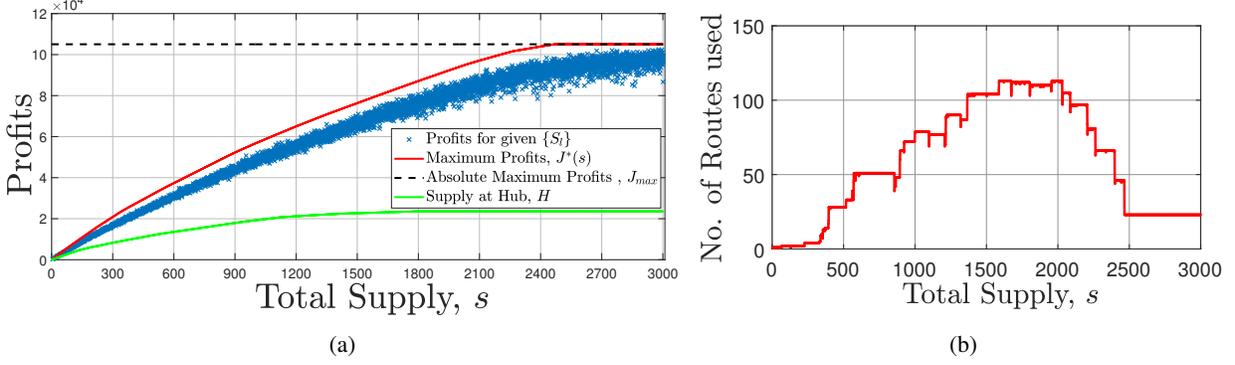


Fig. 4: Simulation results for feed-in problem. **(a)**: The red curve is the maximum profits as a function of total supply. The green curve is the maximum profits, with all supply at H , as a function of total supply s . The blue scatter points are for different supply configurations, with different supplies s . **(b)**: Number of routes utilized for obtaining maximum profits in (a).

4) *Equivalence of Feed-Out and Feed-In Supply Optimization*: We use the directed graph in Figure 1b and the construction in Steps (C1)-(C3) to generate a feed-out problem from the feed-in supply optimization problem. Using route set $\tilde{\mathbf{R}}^-$, we generate the optimal profits for the same instances of total supply as before and compared it with the maximum profits for the feed-in supply optimization. The absolute error is in the range of 10^{-4} while the maximum relative error is 5.66×10^{-6} , which is within numerical tolerance given the solver precision is 10^{-8} and the number of variables are 12695. This verifies the equivalence of the two problems.

B. Microscopic Implementation

In real life, the demands and supplies are indivisible entities, specifically integers. Thus in this section, we implement the microscopic optimization and compare it with the macroscopic optimization. Specifically, in this subsection, we assume that for each node l , $\tilde{d}_l \in \mathbb{N}_0$ and $\tilde{S}_l \in \mathbb{N}_0$ are the total passengers and the number of vehicles, respectively. For simplicity, we assume all the vehicles have the same capacity of N . In the macroscopic problem, as we assumed that a unit demand can be serviced by a unit of supply, in this subsection we assume that 1 unit of demand and supply are N passengers and 1 vehicle, respectively. With this interpretation, we redefine problem (9) with integer constraints. We let $\tilde{f}_r^i(l)$ be the number of people serviced on (r, i, l) and \tilde{f}_r be the number of vehicles taking route r . We also assume that the operator can choose the supply configuration as in Section V. To incorporate that aspect, we assume that the operator has a total supply of s at its disposal. Then the constraints discussed in (8) and (16) are modified to

$$\tilde{F}_l := \sum_{(r,i)} \tilde{f}_r^i(l) \leq \tilde{d}_l, \quad \sum_{l \in V_r^i} \tilde{f}_r^i(l) \leq N \tilde{f}_r, \quad \sum_{r \in \mathbf{R} | O_r=l} \tilde{f}_r \leq \tilde{S}_l, \quad \sum_{l \in V} \tilde{S}_l \leq s, \quad (24)$$

along with the integer constraints on all optimization variables.

In (10), we assumed that $\beta_r^i(l)$ is the operator income per unit demand on the service (r, i, l) . Therefore, the operator income per an individual passenger on the service (r, i, l) is $\beta_r^i(l)/N$. Given this, the operator income from the service (r, i, l) is $\tilde{f}_r^i(l)\beta_r^i(l)/N$. Further, the travelling cost incurred on route r is $\tilde{f}_r c_r$. Therefore, the *microscopic operator profit maximization problem* is

$$\begin{aligned} \max_{\tilde{f}_r^i(l), \tilde{f}_r, \tilde{S}_l} J = & \sum_{(r,i,l) \in \mathbf{W}(\mathbf{R})} \tilde{f}_r^i(l) \beta_r^i(l) / N - \sum_{r \in \mathbf{R}} \tilde{f}_r c_r \\ \text{s.t. } & (24), \quad \tilde{f}_r^i(l), \tilde{f}_r, \tilde{S}_l, \tilde{F}_l \in \mathbb{N}_0. \end{aligned} \quad (25)$$

The corresponding macroscopic problem is the same as (16). We first compare the difference in the optimal values of (25) with $\mathbf{R} = \mathbf{R}_I$ and $\mathbf{R} = \mathcal{R}(\mathbf{R}_I)$ with simulations. We call these the *complete micro problem* and the *reduced micro problem*, respectively. We first show that the reduced route set is also effective for microscopic solutions. We simulated for five different vehicle capacities: 2, 4, 5, 6 and 10. Note that with integer constraints, the problem for the graphs in Figure 1 becomes rather large. Hence, for these set of simulations, we have chosen a

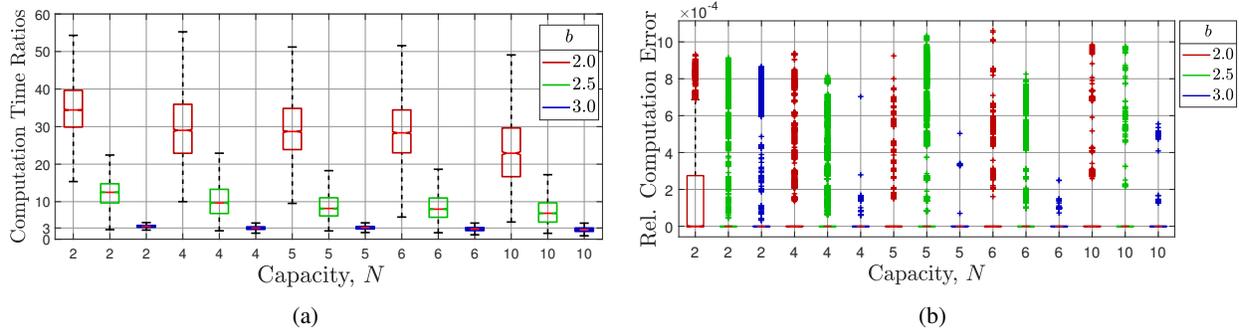


Fig. 5: **(a)** Ratio of computation time for the complete micro problem to the computation time for the reduced micro problem for different b 's, and **(b)** Relative error of objective values between optimal solutions for the complete micro problem and the reduced micro problem on a 12 node graph for different values of b and different capacities. The red, green and blue boxes represent $b = 2, 2.5,$ and 3 respectively.

12 node directed graph with a time window of $T = 25$ and $b = 2, 2.5$ & 3 . For $\mathbf{R} = \mathbf{R}_I$, the number of routes was 11694 and the number of variables was 21127. On the other hand, the problem with $\mathbf{R} = \mathcal{R}(\mathbf{R}_I)$ has 75 routes and 103 variables for $b = 2$, 265 routes and 510 variables for $b = 2.5$, and 1116 routes and 2103 variables for $b = 3$. Gurobi required ≈ 0.35 to 0.5 GB of memory for the reduced micro problems (for all b) while it required ≈ 5 GB of memory for the complete micro problems. The 12-node graph data is given in the supplementary material.

From Figure 5a, we observe that the computation time ratios between the complete micro problem and the reduced micro problem for different capacities is about 25 to 40 times with $b = 2$, 8 to 15 times with $b = 2.5$ and 2 to 3 times for $b = 3$. Also, from Figure 5b we observe that for most simulations, the optimal values returned by the complete and reduced micro problems are very similar. Only in about 4% of the cases the value of the complete micro problem was higher and even then only by about 10^{-4} times relative to the objective value of the reduced micro problem. Since, Gurobi sometimes doesn't converge and returns erroneous values due to lower values of MIP Gap and numerical sensitivity [50], the MIP Gap was set to 0.001 to economize on time while ensuring reliability of results.

Next, we compare the macroscopic problem and the complete microscopic problem for the graphs in Figure 1. We again set time windows of $T = 20$ for Graph I and $T = 30$ for Graph II and $b = 2.5$ and $\alpha = 0.5$ for both the graphs. We chose 10 different demand configurations randomly for both the graphs in Figure 1.

Since the complete microscopic problem is very demanding in terms of both the computation time and memory requirements, we do the comparison in an indirect manner. Notice that the macroscopic problem is a relaxation of both the complete and reduced micro problems. Hence, the solutions to the macroscopic problem and the reduced micro problem provide an upper and lower bound, respectively, on the optimal objective value for the complete micro problem. Thus, by comparing these bounds, we can also indirectly compare the solutions to the macroscopic problem and the complete micro problem.

In general it is expected that with higher values of capacity N , the optimal values of the solutions will be far-off from the solutions obtained from (9) and the supply optimization problem proposed in Section V. Thus, for a successful implementation, the relaxation needs to find optimal solutions which are "close" to the integer optimal.

We first note the actual range of computation times of the reduced micro problem in Figure 6. We observe that the computation times generally tend to increase with N . This is because more branch and bound iterations are necessary to arrive at the integer optimal solution. Next, in Figure 7, we notice that the relative difference between the optimal values of reduced micro problem and the macroscopic problem is quite small, with a median of the order of 10^{-3} . This is a reasonably small value considering the number of variables, the order of the optimal objective value and the computation power utilized for computing these solutions. Recall that the solutions to the macroscopic problem and the reduced micro problem provide upper and lower bounds, respectively, on the optimal objective value in the complete micro problem. Therefore, the relative difference between the complete micro problem and the macroscopic problem is smaller than the errors in Figure 7. Thus, the macroscopic problem is a reasonably good approximation of the reduced micro problem and thereby the complete micro problem as well. We also observed that when the demand is an integer multiple of the supply, there exists an integer solution. Furthermore, the solution was calculated in times similar to the macroscopic problem. We suspect that this behaviour is because of the fact

that a unit of supply can pick-up a unit demand at one location itself and thereby no inter-node mode-sharing is required.

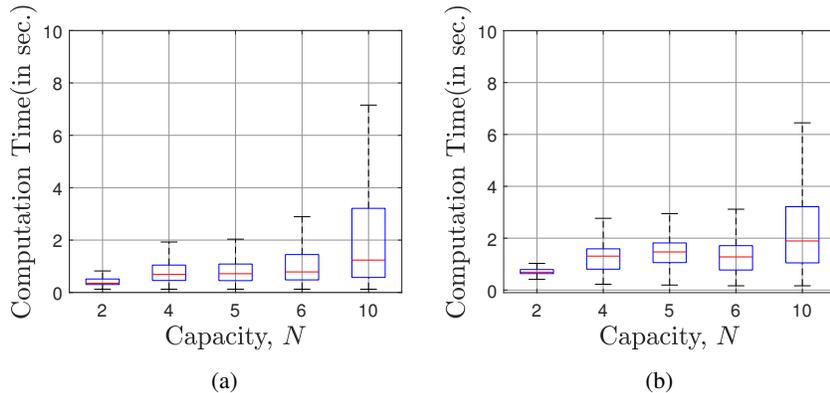


Fig. 6: Computation times of optimal solutions of reduced micro problem for Graph I and II in (a) and (b), respectively.

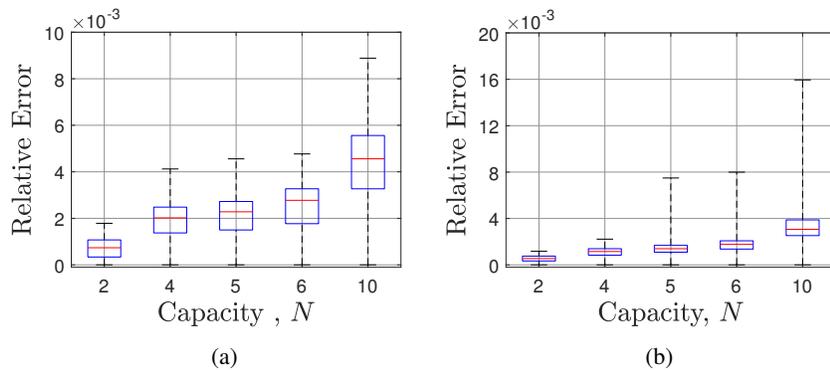


Fig. 7: Relative errors between the optimal objective values of macroscopic and reduced micro problems for Graphs I and II in (a) and (b), respectively.

C. Pre-positioning of Relief Material for Disaster Response

As we discussed in Remark II.1 and in Section IV-B, our model can be used even in the context of routing and planning for disaster response, where the objective is not to maximize monetary profits but to maximize social welfare. Post-cyclone relief efforts are also on an equally massive scale. In this subsection, we consider the problem of planning the routing of relief supplies prior to a cyclone in the state of Odisha.

For the illustrative example here, we chose 12 coastal districts in the state of Odisha that have been severely affected in recent cyclones. The most vulnerable areas are generally low-lying and near the coast as seen in Figure 8. All these districts were put on alert during Cyclone Amphan in 2020. The setup is as follows. The nodes in the graph are either major towns or interchange points of 2 major highways in the districts and the links between the nodes are the major highways between these district locations. We chose the hub node as Bhubaneswar, the capital city of the state of Odisha. We chose the link costs and travel times proportional to the distance, which we created using the *Google My Maps* tool. We present the map and the constructed network in Figure 8. Though the links are marked bi-directional, there are some minor asymmetries. We present the link data in the supplementary material. We look at the problem of transporting of relief material from the hub node to the remaining nodes.

We chose the demand configuration as follows. We assume that the number of affected people is 2% of the total population of each district. Furthermore, we assume that a unit demand refers to relief material for 400 people. We

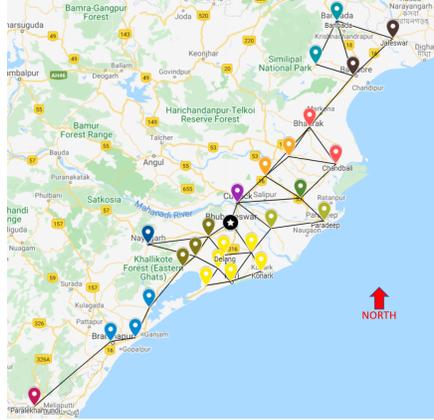


Fig. 8: Map of the graph utilized for Odisha. This map was created with *Google My Maps* tool. Markers of same colour represent places in same district. The Hub (Bhubaneswar) is marked with a black star.

round off the demands to the nearest multiple of 5 and we added 10 extra units as a buffer. We assume that loading of trucks takes 2 hours. We assume that initially, all the supply of trucks is stationed at the hub node. Thus, for multi-legged routes we add a wait time of 2 hours per leg. The main goal here is to maximize the demand met while reducing transportation costs.

With this setup, we present two sets of simulation results. In the first set of simulations, for each node l we chose the value of k_l randomly in $[-4, 0]_{\mathbb{Z}}$. With these settings, we varied the total supply s and the length of the time window T . We show the results on the demand met in Figure 9a. As can be seen, the obvious trend is that for a fixed T more demand can be met given greater supply. Similarly, for each fixed total supply s , longer time-windows mean more demand is met.

In the second set of simulations, we fix the window time $T = 12$ and fix the values of k_l to the same value for all nodes l . Figure 9b shows the simulation results for different values of supply s and different values of k_l . For each fixed supply, as we increase the magnitude of k_l more demand is met. Thus, in general, k_l can be made proportional to the severity of the cyclone in a given region/node l to divert the supplies accordingly.

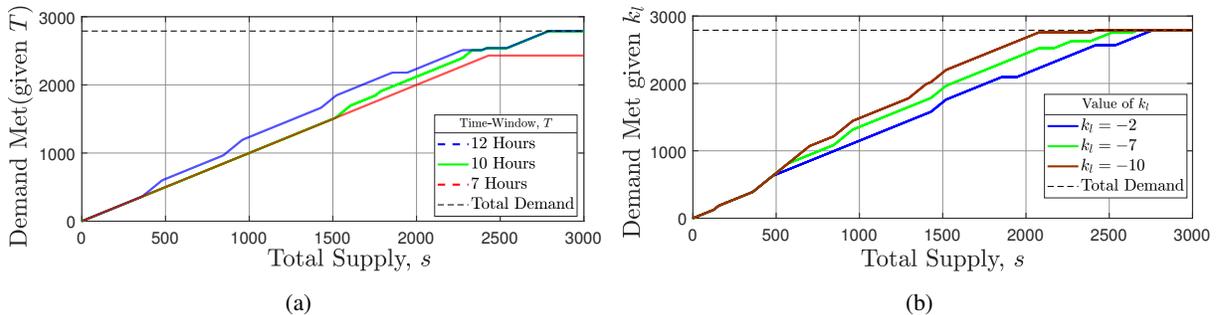


Fig. 9: Simulation results for the Odisha supply prepositioning model. **(a):** Demand met given different window times T . **(b):** Demand met given window time, $T = 12$ hours and different values of k_l .

VII. CONCLUSIONS

In this paper, we proposed a problem of *one-shot* coordination of first mode or last mode transportation service, wherein an operator seeks to maximize its profits or social welfare with routing and allocation, for transporting a known demand to or from a common destination on a network in a given fixed time window. We posed the problem in a macroscopic setting where we considered all supplies and demands as volumes. Using K.K.T. analysis we were able to design an offline (supply and demand independent) method that reduces the complexity of the

online (after supply and demand are revealed) optimization. Then, we considered the feed-in supply optimization problem, analyzed its properties and computed the absolute maximum profits that the operator can earn over all possible supply configurations for a given demand configuration. We showed an equivalence between the feed-in supply optimization problem and the *one-shot feed-out* problem, wherein the operator needs to drop-off people to their destinations from a common origin within a fixed time window. This allows us to directly apply the results and algorithms developed for the feed-in problem. We presented a pricing model and derived necessary conditions for the feeder service to be viable. Finally, we presented several simulation results to illustrate our analytical results. Through simulations, we also demonstrated that the route reduction algorithm that we proposed for the macroscopic formulation is still useful for efficiently computing the solutions to the microscopic formulation, in which the decision variables are all integers. We also explored a realistic scenario of disaster response.

Future work includes improvements in the route reduction algorithm, perhaps making it also dependent on the demand and supply configurations, a more rigorous study of the macroscopic formulation and the route reduction algorithm as a computationally efficient solution method for the microscopic formulation; extension to a multiple time window problem, load balancing of the supply in accordance with the anticipated demand using the insights from the feed-in supply optimization problem, extension to the scenario with uncertainty about supply and demand and finally an integrated coordination of multiple modes of transportation.

VIII. ACKNOWLEDGEMENTS

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APPENDIX

A. List of Frequently Used Symbols

- $G := (V, E)$ denotes the graph. V is the set of vertices (abstraction of the region’s localities) and E is the set of edges (abstraction of the connectivity between the localities).
- (ρ_{lk}, t_{lk}) are positive edge weights for cost and time of travelling across $(l, k) \in E$.
- d_l, S_l are total demand and supply at a node $l \in V$.
- H and T are the destination and length of time window.
- $r := (V_r, E_r)$ is a route defined by a sequence of nodes V_r and sequence of edges E_r in the route set \mathbf{R}_I .
- O_r, D_r are the origin and destination of route r .
- t_r, c_r, θ_r are travel time, per unit travel costs and number of legs on route r , respectively.
- (r, i, l) denotes the *service tuple*, i.e., a service at node l along the i^{th} leg of the route r .

- $\mathbf{R}, \mathcal{R}(\mathbf{R})$ refer to a general route set and its reduced route-set, respectively.
- $\sigma(l, k, \mathbf{R})$ refer to all the *simple routes* in \mathbf{R} .
- $\mathbf{W}(\mathbf{R})$ is the *set of feasible service tuples given a route set* \mathbf{R} .
- $\mathbf{R}_I, \mathbf{R}_O$ are the *feed-in* and *feed-out* route sets respectively.
- f_r represents the *volume of feeders taking a route* r .
- $f_r^i(l)$ are the allocations on a service (r, i, l) .
- k_l is the operational costs for every unit of allocation on a node l .
- $p_r^i(l), \beta_r^i(l)$ denote the *price charged* and the *operator income* per-unit allocation on (r, i, l) , respectively.

B. Proofs of Results on General One-Shot Feeding Problem

1) *Proof of Proposition III.1:* We first introduce the Lagrangian \mathcal{L} for the problem (9),

$$\mathcal{L} = J + \sum_{l \in V} \mu_l \left(\sum_{r,i} f_r^i(l) - d_l \right) + \sum_{r,i} \lambda_r^i \left(\sum_{l \in V_r^i} f_r^i(l) - f_r \right) + \sum_{l \in V} \gamma_l \left(\sum_{r|O_r=l} f_r - S_l \right) - \sum_r \delta_r f_r - \sum_{r,l,i} \delta_r^i(l) f_r^i(l), \quad (26)$$

where $\mu_l, \lambda_r^i, \gamma_l, \delta_r, \delta_r^i(l) \leq 0$ are the KKT multipliers. The stationarity and complementary slackness conditions are

$$\frac{\partial \mathcal{L}}{\partial f_r^i(l)} = \beta_r^i(l) + \mu_l + \lambda_r^i - \delta_r^i(l) = 0 \quad (27a)$$

$$\frac{\partial \mathcal{L}}{\partial f_r} = -c_r - \sum_i \lambda_r^i + \gamma_{O_r} - \delta_r = 0 \quad (27b)$$

$$\mu_l \left(\sum_{r,i} f_r^i(l) - d_l \right) = 0, \quad \lambda_r^i \left(\sum_{l \in V_r^i} f_r^i(l) - f_r \right) = 0 \quad (27c)$$

$$\gamma_l \left(\sum_{r|O_r=l} f_r - S_l \right) = 0, \quad \delta_r^i(l) f_r^i(l) = 0, \quad \delta_r f_r = 0. \quad (27d)$$

(a): In an optimal solution, if $f_r^* > 0$ then $\delta_r^* = 0$. Further as $\gamma_{O_r}^* \leq 0$ and $c_r > 0$, we can use (27b) to obtain

$$\sum_{i=1}^{\theta_r} \lambda_r^{i*} = -c_r + \gamma_{O_r}^* < 0. \quad (28)$$

Thus, we see from (27c) that $\sum_{l \in V_r^i} f_r^i(l)^* = f_r^*$ for at least one leg i in the route r . Hence we must have $f_r^i(l)^* > 0$ for some $i \in [1, \theta_r]_{\mathbb{Z}}$ and $l \in V_r^i$. Now, if $f_r^i(l)^* > 0$ then $\delta_r^i(l) = 0$. Hence condition (27a) implies

$$\beta_r^i(l) = -\lambda_r^{i*} - \mu_l^* \geq -\lambda_r^{i*} \geq 0, \text{ if } f_r^i(l)^* > 0. \quad (29)$$

(b): We prove this by contradiction. Let us assume that there exists some optimal solution such that (11) does not hold for a circular leg i of some $r \in \mathbf{R}$. Then, $\delta := f_r^* - \sum f_r^i(l)^* > 0$ units of flow costs δc_r^i while earning no revenue.

Now consider a route \bar{r} that follows the same sequence of nodes as r but without the i^{th} leg of r . We can construct another feasible solution with $f_{\bar{r}} = f_r^* - \delta$ and $f_{\bar{r}}^i = f_r^* + \delta$ with all the other flows and allocations remaining the same. Now leg i of route r in this solution satisfies (11). Such a solution is feasible and does not lose δc_r^i while earning same revenue. Thus, this solution earns more profit than the optimal solution, which is a contradiction.

(c): Notice that $f_r^i(l)^* \beta_r^i(l) \geq -f_r^i(l)^* \lambda_r^{i*}$ for each (r, i, l) , since if $f_r^i(l) = 0$ then the inequality holds trivially and if $f_r^i(l) > 0$ then the condition (29) holds, which implies

$$\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \beta_r^i(l) \geq - \sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \lambda_r^{i*}.$$

Now, using part (b), we obtain

$$\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \beta_r^i(l) \geq - \sum_{l \in V_r^1} f_r^1(l)^* \lambda_r^{1*} - f_r^* \sum_{i=2}^{\theta_r} \lambda_r^{i*}.$$

From the K.K.T. conditions, we either have $\lambda_r^{1*} = 0$ or $\sum_{l \in V_r^1} f_r^1(l)^* = f_r^*$. In either case, we have

$$\sum_{i=1}^{\theta_r} \sum_{l \in V_r^i} f_r^i(l)^* \beta_r^i(l) \geq -f_r^* \sum_{i=1}^{\theta_r} \lambda_r^{i*} \geq f_r^* c_r,$$

where the second inequality follows from (28).

(d): Now due to Part **(c)**, if $f_r^* > 0$ then $\forall i > 1, \exists l \in V_r^i$ such that $f_r^i(l)^* > 0$. In any feasible solution, for $i > 1$, if $\beta_r^i(l) < c_r^i$ and $f_r^i(l) > 0$ then we can construct another feasible solution in which $f_r^i(l) = 0$ keeping all other node allocation variables unchanged. The value of the objective is *strictly* more with such a new solution. Thus, $\beta_r^i(l) \geq c_r^i$ if $f_r^i(l)^* > 0$ for $i > 1$ in an optimal solution of Problem (9). If $i = 1$ and leg 1 of route r is circular then according to a similar argument as above, we must again have at least one $l \in V_r^1$ such that $f_r^1(l)^* > 0$ and for each such $l, \beta_r^1(l) \geq c_r^1$.

(e): In part **(c)**, by setting $\theta_r = 1$, we get

$$\sum_{l \in V_r^1} f_r^1(l)^* \beta_r^1(l) \geq f_r^* c_r.$$

Similarly, setting $\theta_r = 1$ in (28) gives us $\lambda_r^1 \leq -c_r < 0$, which along with (27c) means that in any optimal solution, constraint (8b) is a strict equality for single legged routes. Therefore, we have $\sum_{l \in V_r^1} (\beta_r^1(l) - c_r) f_r^1(l)^* \geq 0$. Now, part **(e)** follows in the same way as part **(d)**. \square

C. Proofs of Results on Best Transportation Parameters

1) *Proof of Lemma IV.2:* Consider an arbitrary node $l \in V$ and a service tuple (r, i, l) such that $l \in V_r^i$. Then, from (13) and (10) notice that

$$\beta_r^i(l) - c_r^i = g_l(b) - \alpha t_r^i(l) - c_r^i - k_l \leq g_l(b) - g_l(1) - k_l$$

where $g_l(b)$ is as defined in (15). The inequality follows from the fact that a leg is also single legged route and therefore $g_l(1)$ lower bounds $\alpha t_r^i(l) - c_r^i$. Thus it is easy to see that $\beta_r^i(l) - c_r^i \geq 0$ and hence $\mathcal{R}(\mathbf{R}) \neq \emptyset$ only if $g_l(b) \geq g_l(1) + k_l$. Now, we show that if $\exists l \in V$ such that $g_l(b) \geq g_l(1) + k_l$, then $\mathcal{R}(\mathbf{R}) \neq \emptyset$. We show that $r(l, 1)^* \in \mathcal{R}(\mathbf{R})$ for this case. Thus, considering $q = r(l, 1)^*$, as per Table I, we have

$$\beta_q^1(l) - c_q^1 = g_l(b) - \alpha t_q^1(l) - k_l - c_q^1 = g_l(b) - (\alpha t_q + c_q + k_l) = g_l(b) - (g_l(1) + k_l) \geq 0.$$

Thus, $q \in \mathbf{R}_1$ which implies $\mathcal{R}(\mathbf{R}) \neq \emptyset$, if $g_l(b) \geq g_l(1) + k_l$.

Now, we know from Remark IV.1 that $g_l(b)$ is an increasing function of b . Thus, for $k_l \geq 0, b \geq 1$ is necessary to ensure $\mathcal{R}(\mathbf{R}) \neq \emptyset$. Again, for $k_l < 0$ for some $l \neq H$, and $b = 1, g_l(b) - g_l(1) - k_l \geq 0$, following the discussion above. \square

2) *Proof of Proposition IV.3:* Consider a route $r \in \mathcal{R}(\mathbf{R})$ with two distinct legs with feasible services. Note that such a route, has atleast one leg, i with $O_r^i = D_r^i = H$. Also, route $r \in \mathbf{R}_2$ (see (12c)) as $\theta_r \geq 2$. Now, let \bar{r} be the route that is same as leg i of route r . Thus, $O_{\bar{r}} = D_{\bar{r}} = H, \theta_{\bar{r}} = 1$ and $w_{\bar{r}}^1 \geq 0$, which implies $\bar{r} \in \mathcal{R}(\mathbf{R})$ as $\bar{r} \in \mathbf{R}_1$ (see (12b)). Hence, from (13) and (10), $w_{\bar{r}}^1 = \max_{l \in V_{\bar{r}}^1} \{\beta_{\bar{r}}^1(l) - c_{\bar{r}}\} \geq 0$ implies

$$g_l(b) \geq \alpha t_{\bar{r}}^1(l) + c_{\bar{r}} + k_l, \text{ for some } l \in V_{\bar{r}}, l \neq H, \quad (30)$$

where we have used the fact that $g_H(b) = 0$ for all $b \geq 0$ and $k_H > 0$, which imply that $l \neq H$. Now, note that for

$$\begin{aligned} \alpha t_{\bar{r}}^1(l) + c_{\bar{r}} &= \alpha t_{\bar{r}}^1(l) + c_{\bar{r}}(H, l) + c_{\bar{r}}(l, H) \\ &\geq \begin{cases} g_l(1) + c^*(H, l), & \text{if } \mathbf{R} = \mathbf{R}_I \\ g_l(1) + c^*(l, H), & \text{if } \mathbf{R} = \mathbf{R}_O \end{cases} \\ &= g_l(1) + c^*(D, O) \end{aligned}$$

where we have split the route cost $c_{\bar{r}}$ into $c_{\bar{r}}(H, l)$, the cost to go from H to l on route \bar{r} , and $c_{\bar{r}}(l, H)$, the cost to go back from l to H . Therefore, from (30), $g_l(b) \geq g_l(1) + c^*(D, O) + k_l$, thus proving claim (a).

Now using (15), we get $g_l(b) = \alpha t_{r(l,b)^*} + b c_{r(l,b)^*}$ and $g_l(1) = \alpha t_{r(l,1)^*} + c_{r(l,1)^*}$. Therefore, claim (b) is true.

Next, from Lemma IV.2, we know that if $k_l > 0$ for all $l \in V$ then $\mathcal{R}(\mathbf{R})$ is non-empty only if $b > 1$. Then, notice from (14) and (15) that $c_{r(l,0)^*} \geq c_{r(l,1)^*} \geq c_{r(l,b)^*}$ and $t_{r(l,b)^*} \leq t_{r(l,\infty)^*}$. Using these facts we obtain claim (c). Finally, as $c_{r(l,b)^*}$ is a non-increasing function of b we conclude that $b_l^* \geq \hat{b}_l^*$. \square

D. Proofs of Results on Feed-in Supply Optimization

1) *Proof of Proposition V.1:* Part (a) is true because Theorem III.2 applies for every possible supply configuration. We prove the remaining parts by contradiction.

(b): Let us assume there exists an optimal solution, which we denote using a superscript $*$, in which $f_r^* > 0$ for a route $r \in \mathcal{R}(\mathbf{R}_I)$ such that $f_r^1(O_r)^* < f_r^*$. Then, there are the following two cases. (i) $f_r^1(O_r)^* > 0$ and $f_r^i(l)^* = 0$ for all other legs and nodes (i, l) on route r ; and (ii) there is a pair $(i_1, l_1) \in [1, \theta_r]_{\mathbb{Z}} \times V_r$ with $(i_1, l_1) \neq (1, O_r)$ and $f_r^{i_1}(l_1)^* > 0$. Note that Proposition III.1(b) implies that scenario (i) may occur only if the route is simple ($\theta_r = 1$).

(i) In this case, clearly the original solution cannot be optimal because $f_r^* - f_r^1(O_r)^*$ volume of vehicles simply traverse the route without serving any demand, thus incurring a non-zero cost while earning nothing.

(ii) Without loss of generality let (i, l_1) be the leg, node pair other than $(1, O_r)$ that first appears in the sequence given by the route r such that $f_r^i(l_1)^* > 0$. Then consider the route \bar{r} , which is the sub-route of r formed by excluding all nodes in r that occur prior to (i, l_1) so that $o_{\bar{r}} = l_1$. Thus, for each leg and node pair (j, m) of the route \bar{r} there is a unique leg and node pair (k, l) of route r such that $(j - i + 1, m) = (k, l)$ and moreover they appear in the same order. Now consider a solution $f_r = f_r^1(O_r) = f_r^1(O_r)^*$ and $f_{\bar{r}} = f_{\bar{r}}^* + (f_r^* - f_r^1(O_r)^*)$ and such that $f_{\bar{r}}^j(m) + f_{\bar{r}}^k(l) = f_{\bar{r}}^j(m)^* + f_{\bar{r}}^k(l)^*$ for every $m \in V_{\bar{r}}^i$ and for each leg j of route \bar{r} . This solution is feasible and earns higher profits than the original solution as the flow $(f_r^* - f_r^1(O_r)^*)$ does not have to traverse the sequence of nodes from $(1, O_r)$ to (i, l_1) and the node allocations are unchanged. This again contradicts the assumption that the original solution is optimal. Thus, for each $r \in \mathcal{R}(\mathbf{R}_I)$, $f_r^1(O_r)^* = f_r^*$. As a consequence of this fact and Proposition III.1(b), we also have (17). Also, $d_H = 0$ implies $f_r^* = 0, \forall r$ s.t. $O_r = H$.

(c): Assume an optimal solution with $f_r^* > 0$, for a $r \in \mathcal{R}(\mathbf{R}_I)$ with $\beta_r^1(O_r) < c_r^1$. Using (b), we know $f_r^1(O_r)^* = f_r^* > 0$. However, one could set $f_r^1(O_r)^* = 0$ and move the supply on O_r as in part (b). Then, the so constructed solution would again earn higher profits, which contradicts the assumption that the original solution is optimal. Consequently, with Proposition III.1(d), $\exists l \in V_r^i, \text{ s.t. } f_r^i(l)^* > 0, \beta_r^i(l) \geq c_r^i, \forall i \in [1, \theta_r]_{\mathbb{Z}}$.

(d): Suppose that a node l violates the property in (A4) and yet $f_r^i(l)^* > 0$ for some route r and a leg i . Then by part (c) we must have $\beta_r^i(l) - c_r^i \geq 0$. Now, given the service tuple (r, i, l) , consider a simple route q , which is the sub-route of route r from the last visit to node l in leg i to H in that leg. As a result, $o_q = l$ and $q \in \mathbf{R}_I$. Now, notice that

$$\beta_q^1(l) - c_q^1 \geq \beta_r^i(l) - c_r^i \geq 0,$$

since q is a sub-route of the leg i of route r . This contradicts the assumption that l violates the property in (A4).

(e): One can construct another route q from r by avoiding the cycle. Again, moving the supply to this route (in a manner similar to the previous parts) earns more profits.

(f): In the scenario $s \leq \sum_l d_l$, the key observation is that the full demand cannot be served by simple routes. Thus, if $\sum_{r|O_r=l} f_r^* < S_l$ for some node $l \in V$ then there is some unused supply. Such redundant supply could

potentially be used to serve more demand either at node l or moved to a different node \bar{l} to meet the demand there with simple routes. Assumption (A4) implies that there exist simple routes to which if the redundant supply is reallocated then the profits are higher. This contradicts the assumption that the original solution is optimal. Thus, in every optimal solution, we have

$$S_l = \sum_{r|O_r=l} f_r^* \leq F_l^* \leq d_l, \quad \forall l \in V,$$

where the second inequality is just one of the constraints in the optimization problem. \square

2) *Proof of Lemma V.2:* By the definition of $\mathcal{S}(l)$ in (20), $\beta_r^1(l) - c_r > \beta_q^1(l) - c_q, \forall r \in \mathcal{S}(l)$ and $\forall q \in (\sigma(l) \setminus \mathcal{S}(l))$. Every other route \bar{r} either originates from a different location or has multiple legs and in each case the leg/route cost is higher and the operator incomes are lower. Therefore, $\exists q \in (\sigma(l) \setminus \mathcal{S}(l))$ for which $\beta_q^1(l) - c_q > \beta_{\bar{r}}^i(l) - c_{\bar{r}}^i$. Hence, $\beta_r^1(l) - c_r > \beta_{\bar{r}}^i(l) - c_{\bar{r}}^i$ for all $r \in \mathcal{S}(l)$ and $\bar{r} \notin \mathcal{S}(l)$. Further, for $\forall r \in \mathcal{S}(l)$ and $\forall q \in (\sigma(l) \setminus \mathcal{S}(l))$, the perceived costs satisfy

$$(\alpha t_r + c_r) - (\alpha t_q + c_q) = (\alpha t_r + c_r - g_l + 1) - (\alpha t_q + c_q - g_l + 1) = -(\beta_r^1(l) - c_r) + (\beta_q^1(l) - c_q) < 0, \quad (31)$$

where we have used (13) and (10) and the fact that $o_z = l$ for all $z \in \sigma(l)$, which implies that the route traversal time $t_z = t_z^1(l)$. This proves the result. \square

3) *Proof of Lemma V.5:* As for both the problems, the time window, best travel cost and the best travel time are same, it therefore suffices to show that $t_{\bar{r}}^{\theta_r - i + 1}(l) = \hat{t}_r^i(l)$. Note that the i^{th} leg of r is same as the $(\theta_r - i + 1)^{\text{th}}$ leg of \bar{r} in reverse. Also, $T - t_{\bar{r}}^{\theta_r - i + 1}(l)$ is the last possible pick-up time along $(\bar{r}, \theta_r - i + 1, l)$, based on the observations of (13) which implies that $T - t_{\bar{r}}^{\theta_r - i + 1}(l)$ is the first possible drop-off time for the reverse route r along the service tuple (r, i, l) . \square